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THE QUARTIC EFFECTIVE ACTION OF THE HETEROTIC STRING AND SUPERSYMMETRY

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We present the supersymmetric quartic effective action for the heterotic string which follows from the supersymmetrization of the Yang–Mills and Lorentz Chern–Simons forms. This includes all bosonic terms in the action, and all bosonic contributions to the supersymmetry transformation rules, thereby giving all terms to this order which are relevant for the study of compactification scenarios with unbroken supersymmetry.

1. Introduction

It is a major challenge in superstring theory to relate the intrinsic properties of strings to particle physics. One approach to this problem is to investigate the low-energy effective action [1], in which string effects should appear in the form of interaction terms which are absent in more conventional supergravity theories. In this paper we consider the low-energy limit of the ten-dimensional heterotic string [2], which corresponds to ten-dimensional supergravity coupled to Yang–Mills. Application to physics implies that compactification to four dimensions is required, and for phenomenological reasons a remaining $N=1$ supersymmetry in four dimensions would be preferred (for a review of the phenomenology of $N=1$ supergravity, see e.g. ref. [3]). One way to investigate the possibilities for such compactifications is to consider the ten-dimensional effective action. This effective action is therefore a crucial ingredient in phenomenological applications of string theories.

The purpose of this paper is to obtain all contributions to the effective action up to and including terms quartic in the Yang–Mills and gravitational curvatures, and

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to obtain the transformation rules under local supersymmetry of the fields. Thereby we obtain all relevant terms for the study of compactification to this order. These terms have been collected in appendix A. The main body of this paper deals with the derivation of these results. The reader who is only interested in the final answer is referred to appendix A which can be studied independently.

Essentially four methods have been employed thus far to obtain information about the effective action. String amplitude calculations [4–8] provide information about the bosonic part of the effective action, and have the advantage that the string aspects are implicitly taken care of. On the other hand, it is difficult to incorporate fermions in this approach, and supersymmetry therefore remains unclear. Although calculations can be performed for tree-level [4,5] and one-loop amplitudes [6–8], extensions to higher loops are extremely difficult. Nevertheless, some results for two-loop amplitudes have been recently obtained [9]. Another approach is through the calculation of loop corrections in supersymmetric sigma-models [10–12]. The requirement that the β-function vanishes should determine the equations of motion of bosonic background fields. These equations of motion then determine the desired effective action. Also in this approach the inclusion of fermions and supersymmetry is nontrivial. One way to avoid this problem is by taking supersymmetric field theory as a starting point. The obvious problem is then to relate to the string aspects. Both superspace [13,14],[15,16] and Noether methods [17–19] have been applied so far. Superspace seems to have a natural connection with strings through e.g. the Green–Schwarz superstring [20]. However, superspace methods are technically involved, and not very explicit in view of the applications we have in mind. The Noether method, which we employ in this paper, appears at first sight rather primitive. Nevertheless, its explicit nature as well as a number of tricks which we will explain below and in the next sections, make it a viable approach to the construction of effective actions.

The main guideline in our construction of the quartic effective action will be the assumption that the Yang–Mills and gravitational contributions to the effective action should appear symmetrically. The first place where this symmetry is important is in the construction of the quadratic effective action, i.e. the action that contains quadratic terms in the Yang–Mills and gravitational curvatures. This quadratic action includes Yang–Mills and Lorentz Chern–Simons forms. The Yang–Mills Chern–Simons form was obtained in the coupling of $d=10$ supersymmetric Yang–Mills theory to supergravity [21–23]. The Lorentz Chern–Simons form plays a crucial role in the cancellation of anomalies in the $d=10$ Einstein–Yang–Mills theory [24]. It breaks local supersymmetry however, and much effort has been devoted to the construction of its supersymmetric version in ten dimensions.

A rather simple method to construct the quadratic effective action that includes both the Yang–Mills and the Lorentz Chern–Simons form is to employ a symmetry that exists between the Yang–Mills and supergravity fields in ten dimensions [25].

This symmetry between Yang–Mills and supergravity also exists in six dimensions and has been used to construct a supersymmetric R^2-action for $d=6$ conformal supergravity [26]. It has also been obtained in superspace [16, 27], and has been used in the construction of $d=10$ R^2-actions in that context [16]. Since this relation is a crucial ingredient of the present work, let us briefly discuss the essential point.

The spin-connection $\omega_\mu{}^{ab}$ of d-dimensional gravity plays the role of an SO($d-1,1$) gauge field, gauging the local Lorentz transformations which are a part of the gauge symmetries of supergravity. At first sight, this suggests that in constructing an R^2-action one should take the Yang–Mills F^2-action, and replace everywhere $F(A)$ by $R(\omega)$. Although this looks promising at first sight, the two gauge fields A (Yang–Mills) and ω (Lorentz) do not have the same behaviour under supersymmetry transformations. In $d=10$ however the way to continue is rather clear: one should not identify ω with an SO(9,1) gauge field, but rather an appropriate combination of ω and H, where H is the field strength of the antisymmetric gauge field, $B_{\mu\nu}$, of $d=10$ supergravity. With a suitably chosen basis for the supergravity fields the identification can then be made and the construction of the R^2-action becomes trivial [25].

This paper is organized as follows. In sect. 2 we give a short overview of the construction of the quadratic action [25]. More details are given in appendix B. We use coupling constants α and β to distinguish between different sectors (quadratic, cubic, quartic) of the action. With g being the Yang–Mills coupling constant, we use $\beta = 1/g^2$. For the SO(9,1) multiplet which represents the supergravity sector we use an analogous coupling constant α. It has the same dimension as β, and is proportional to α', the inverse of the string tension. The quadratic effective action is then of the form $\alpha R^2 + \beta F^2$.

In sect. 3 we discuss the cubic $\alpha^2 R^3 + \alpha\beta R F^2$-action. We find that there are no purely bosonic terms in this action, in agreement with string amplitude calculations. Nevertheless, sect. 3 is a key section to this paper. In particular we prove a lemma, which essentially says that the variation of the αR^2-action gives only terms which are proportional to the equation of motion of the supergravity fields at $\mathrm{O}(\alpha^0)$, i.e. the equations of motion following from the R-action. This lemma is the key to the remainder of the paper, since it is used in many places to cancel contributions to the variation of the action, also in higher orders. Finally, although no terms in the action are generated which are relevant for compactification, we are forced to modify the supersymmetry transformation rules of the supergravity fields. These are crucial for compactification, and are among the terms collected in appendix A.

Sect. 4 is devoted to the construction of the quartic $\alpha^3 R^4 + \alpha^2\beta R^2 F^2 + \alpha\beta^2 F^4$ action. We make use of the results of string amplitude calculations to make an ansatz for the bosonic part of the quartic action. We then determine by a Noether-method calculation the leading terms of the quartic action, and in particular all terms which are relevant to compactification scenarios. Again new variations of supergravity, and now also of Yang–Mills fields, are required. Our results do not coincide in all detail with those of string calculations [5]. One should keep in mind, however, that there is a certain ambiguity in these higher order invariants. One always has the freedom to redefine the supergravity and Yang–Mills fields order by order, and this in general will modify the effective action. Therefore, in comparing different results, one should only consider those terms which are not affected by such redefinitions.

In this paper we obtain the part of the quartic effective action which follows from the supersymmetrization of the Yang–Mills and Lorentz Chern–Simons forms. This does not yield the complete quartic action. Both string amplitude and sigma-model calculations uncover another R^4 term [4,11], which does not have a Yang–Mills counterpart and which we do not construct here. At the linear level the supersymmetrization of this term is given by superspace techniques [28], and it is probably in that context that its structure is most conveniently described.

In sect. 5 we give our conclusions and outline some approaches to the remaining problems. Appendix A contains all terms in the quartic effective action and transformation rules which are relevant for compactification. Appendix B gives more details of the construction of the quadratic effective action. In particular it contains all higher order fermionic terms in the action and transformation rules.

2. The O(α) and O(β) terms in the effective action

In this section we will review the construction of the R^2-action [25]. The essential ingredient in this construction is the formulation of $d=10$ supergravity as an SO(9,1) Yang–Mills multiplet. This identification is crucial for the extensions to invariants containing higher powers of R as well. This we will discuss in later sections of this paper.

The complete supersymmetrization of the Yang–Mills Chern–Simons form has of course been known for a long time [23]: it is the action of a Yang–Mills multiplet coupled to $d=10$ supergravity. In component form the supersymmetrization to O(α) of the Lorentz Chern–Simons form, the R^2-action, was obtained in ref. [25]. In the present recapitulation of that work, we mainly want to set the stage for the calculation of higher orders in α, and therefore it is not very useful nor illuminating to repeat all details here. In particular, in the higher-order calculations we will no longer keep track of contributions to the action which are quartic in fermions, nor will we present the variations which determine such terms. Accordingly, in the present discussion we will impose the same restriction, and present only leading terms in the action and transformation rules. In appendix B we gather the complete result to O(α) and O(β), including a number of details which were not presented in ref. [25].

The transformation rules of the coupled $d=10$ supergravity and Yang–Mills fields are to lowest order ($\delta_{\alpha^n}(\delta_{\beta^n})$ indicates contributions proportional to $\alpha^n(\beta^n)$,

δ_0 corresponds to the leading terms, independent of α and β):

$$\delta_0 e_\mu{}^a = \tfrac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu, \qquad \delta_0\psi_\mu = \left(\partial_\mu - \tfrac{1}{4}\Omega_{\mu+}^{ab}\Gamma_{ab}\right)\epsilon,$$

$$\delta_0 B_{\mu\nu} = \tfrac{1}{2}\sqrt{2}\,\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]}, \qquad \delta_0\lambda = -\tfrac{3}{8}\phi^{-1}\not{\partial}\phi\epsilon + \tfrac{1}{8}\Gamma^{abc}\epsilon\hat{H}_{abc}, \tag{2.1}$$

$$\phi^{-1}\delta_0\phi = -\tfrac{1}{3}\sqrt{2}\,\bar{\epsilon}\lambda; \qquad \delta_0 A_\mu = \tfrac{1}{2}\bar{\epsilon}\Gamma_\mu\chi, \qquad \delta_0\chi = -\tfrac{1}{4}\Gamma^{ab}\epsilon\hat{F}_{ab}. \tag{2.2}$$

The coupling induces a number of transformations of $O(\beta)$. In particular, the variation of $B_{\mu\nu}$ reads

$$\delta_\beta B_{\mu\nu} = -\beta\sqrt{2}\ \mathrm{tr}\left\{A_{[\mu}\delta_0 A_{\nu]}\right\}. \tag{2.3}$$

In (2.1, 2.2) we have included a number of $O(\beta)$ terms in (super)covariant curvatures. The supercovariant field strength $\hat{H}$ of the two-index tensor gauge field $B_{\mu\nu}$ is defined as

$$\hat{H}_{\mu\nu\rho} \equiv H_{\mu\nu\rho} - \tfrac{1}{4}\bar{\psi}_{[\mu}\Gamma_\nu\psi_{\rho]},$$

$$H_{\mu\nu\rho} \equiv \partial_{[\mu}B_{\nu\rho]} - \beta\sqrt{2}\ \mathrm{tr}\left\{A_{[\mu}\partial_\nu A_{\rho]} - \tfrac{1}{3}A_{[\mu}A_\nu A_{\rho]}\right\}, \tag{2.4}$$

and contains the β-dependent Yang–Mills Chern–Simons form. In the variation of the gravitino in eq. (2.1) we have introduced the combination

$$\Omega_{\mu\pm}^{ab} \equiv \omega_\mu{}^{ab}(e,\psi) \pm \tfrac{3}{2}\sqrt{2}\,\hat{H}_\mu{}^{ab}. \tag{2.5}$$

The transformations (2.1–2.3) leave the action $\mathscr{L} = \mathscr{L}(R) + \mathscr{L}(F^2)$ invariant, where [22, 23]

$$\begin{aligned}\mathscr{L}(R) = e\phi^{-3}\Big\{&-\tfrac{1}{2}R(\omega(e)) - \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} + \tfrac{9}{2}\left(\phi^{-1}\partial_\mu\phi\right)^2\\
&-\tfrac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\mathscr{D}_\nu(\omega(e))\psi_\rho + 2\sqrt{2}\,\bar{\lambda}\Gamma^{\mu\nu}\mathscr{D}_\mu(\omega(e))\psi_\nu + 4\bar{\lambda}\not{\mathscr{D}}(\omega(e))\lambda\\
&+3\sqrt{2}\,\bar{\psi}_\mu\Gamma^\nu\Gamma^\mu\lambda\left(\phi^{-1}\partial_\nu\phi\right) - \tfrac{3}{2}\bar{\psi}_\mu\Gamma^\mu\psi_\nu\left(\phi^{-1}\partial^\nu\phi\right)\\
&+\tfrac{1}{16}\sqrt{2}\,H^{\rho\sigma\tau}\left[\bar{\psi}_\mu\Gamma^{[\mu}\Gamma_{\rho\sigma\tau}\Gamma^{\nu]}\psi_\nu + 4\sqrt{2}\,\bar{\psi}_\mu\Gamma^\mu{}_{\rho\sigma\tau}\lambda - 8\bar{\lambda}\Gamma_{\rho\sigma\tau}\lambda\right]\Big\},\end{aligned} \tag{2.6}$$

$$\begin{aligned}\mathscr{L}(F^2) = e\phi^{-3}\beta\ \mathrm{tr}\Big\{&-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\bar{\chi}\not{\mathscr{D}}(\omega(e),A)\chi\\
&-\tfrac{1}{4}\bar{\chi}\Gamma^\mu\Gamma^{\nu\rho}F_{\nu\rho}\left(\psi_\mu + \tfrac{1}{3}\sqrt{2}\,\Gamma_\mu\lambda\right) + \tfrac{1}{16}\sqrt{2}\,\bar{\chi}\Gamma^{\mu\nu\rho}\chi H_{\mu\nu\rho}\Big\}.\end{aligned} \tag{2.7}$$

The formulae presented here differ from those given in ref. [22]. This is due to a number of redefinitions in the fields and transformation rules. The precise correspondence between (2.1–2.7) and [22] can be found in ref. [25]. The reason for these redefinitions is that we want to bring the supergravity multiplet to a form in which it can be identified with an SO(9,1) Yang–Mills multiplet. The present basis allows such an identification. To see this, note that the leading contribution to the transformation rule of $\Omega_{\mu-}^{\ \ ab}$ is

$$\delta_0 \Omega_{\mu-}^{\ \ ab} = \tfrac{1}{2}\bar{\epsilon}\Gamma_\mu \psi^{ab}, \tag{2.8}$$

where ψ^{ab} is the supercovariant gravitino curvature

$$\psi_{\mu\nu} \equiv \mathscr{D}_\mu(\Omega_+)\psi_\nu - \mathscr{D}_\nu(\Omega_+)\psi_\mu . \tag{2.9}$$

This implies that $\Omega_{\mu-}^{\ \ ab}$ and ψ^{ab} transform as (2.2), i.e. form an SO(9,1) Yang–Mills multiplet. Indeed, a simple calculation shows that the transformation rule of ψ^{ab} is given by

$$\delta_0 \psi^{ab} = -\tfrac{1}{4}\Gamma^{\mu\nu}\epsilon \hat{R}_{\mu\nu}^{\ \ ab}(\Omega_-) . \tag{2.10}$$

As one can see from the more detailed formulae in appendix B, the identification of the multiplet consisting of $\Omega_{\mu-}^{\ \ ab}$ and ψ^{ab} as an SO(9,1) Yang–Mills multiplet coupled to supergravity holds true also when terms of higher order in the fermions are taken into account.

Therefore an invariant of the form $\mathscr{L}(R) + \mathscr{L}(R^2)$ can be written in complete analogy with (2.6 and 2.7):

$$\begin{aligned}\mathscr{L}(R^2) = e\phi^{-3}\alpha\Big\{ & -\tfrac{1}{4}R^{\mu\nu ab}(\Omega_-)R_{\mu\nu}^{\ \ ab}(\Omega_-) - \tfrac{1}{2}\bar{\psi}^{ab}\not{\mathscr{D}}(\omega(e),\Omega_-)\psi_{ab} \\ & -\tfrac{1}{4}\bar{\psi}^{ab}\Gamma^\mu\Gamma^{\nu\rho}R_{\nu\rho}^{\ \ ab}(\Omega_-)\Big(\psi_\mu + \tfrac{1}{3}\sqrt{2}\,\Gamma_\mu\lambda\Big) + \tfrac{1}{16}\sqrt{2}\,\bar{\psi}^{ab}\Gamma^{\mu\nu\rho}\psi_{ab}H_{\mu\nu\rho}\Big\} .\end{aligned} \tag{2.11}$$

This requires an additional transformation of the field $B_{\mu\nu}$, analogous to eq. (2.3):

$$\delta_\alpha B_{\mu\nu} = -\alpha\sqrt{2}\,\Omega_{[\mu-}^{\ \ \ ab}\delta_0\Omega_{\nu]-}^{\ \ \ ab}, \tag{2.12}$$

and a corresponding redefinition of the curvature $\hat{H}$, which now reads:

$$\begin{aligned}\hat{H}_{\mu\nu\rho} \equiv {} & \partial_{[\mu}B_{\nu\rho]} - \tfrac{1}{4}\bar{\psi}_{[\mu}\Gamma_\nu\psi_{\rho]} - \beta\sqrt{2}\ \mathrm{tr}\Big\{A_{[\mu}\partial_\nu A_{\rho]} - \tfrac{1}{3}A_{[\mu}A_\nu A_{\rho]}\Big\} \\ & -\alpha\sqrt{2}\,\Big\{\Omega_{[\mu-}^{\ \ \ ab}\,\partial_\nu\Omega_{\rho-]}^{\ \ \ ab} - \tfrac{2}{3}\Omega_{[\mu-}^{\ \ \ ab}\Omega_{\nu-}^{\ \ ac}\Omega_{\rho]-}^{\ \ \ cb}\Big\} .\end{aligned} \tag{2.13}$$

Consider now the action $\mathscr{L}(R)+\mathscr{L}(R^2)+\mathscr{L}(F^2)$, with eqs. (2.6), (2.7) and (2.11). It is the action of a $\mathrm{G}\times\mathrm{SO}(9,1)$ Yang–Mills multiplet coupled to supergravity. However, there is a difference in interpretation between (2.7) and (2.11). We want to interpret (2.11) as a gravitational R^2-action, and not as an SO(9,1) Yang–Mills action in which the Yang–Mills fields are independent of the supergravity fields. Therefore we must consider the possibility that the transformation rules (2.8) and (2.10) are modified because of eqs. (2.3) and (2.12), the additional transformation rules in the supergravity sector. Indeed, these modifications to $\delta B_{\mu\nu}$ imply the following $\mathrm{O}(\alpha)$ and $\mathrm{O}(\beta)$ transformations:

$$(\delta_\alpha+\delta_\beta)\Omega_{\mu-}^{\ ab}=\tfrac{3}{2}\bar{\epsilon}\Gamma_{[\mu}X_{ab]}\,,\qquad (\delta_\alpha+\delta_\beta)\psi^{ab}=\tfrac{3}{4}\Gamma_{cd}\epsilon T^{abcd}\,, \tag{2.14}$$

$$\text{where}\quad X_{ab}\equiv\alpha\hat{R}_{ab}{}^{cd}(\Omega_-)\psi^{cd}+\beta\operatorname{tr}\hat{F}_{ab}\chi\,,$$

$$T_{abcd}\equiv\alpha\hat{R}_{[ab}{}^{ef}(\Omega_-)\hat{R}_{cd]}{}^{ef}(\Omega_-)+\beta\operatorname{tr}\hat{F}_{[ab}\hat{F}_{cd]}\,. \tag{2.15}$$

Therefore, invariance of $\mathscr{L}(R)+\mathscr{L}(R^2)+\mathscr{L}(F^2)$ holds only to $\mathrm{O}(\alpha)$ and $\mathrm{O}(\beta)$.

This concludes the construction of the R^2-action. Note that the only transformations which break the exact invariance are (2.14). This is guaranteed by the known invariance of (2.6) and (2.7), in which higher order terms in β are already taken into account. Thus, to obtain the variation of the action in the next order, we only have to use the variations (2.14) of $\Omega_{\mu-}^{\ ab}$ and ψ^{ab} in eqs. (2.11) and (2.6). As we shall see in the next sections, the relatively simple structure of eqs. (2.14) and (2.15) allows us to determine higher-order invariants as well. The symmetry between G (Yang–Mills) and SO(9,1) (supergravity) sectors in (2.14) and (2.15) will play an important role in these calculations. It is due to the fact that both the $\mathrm{O}(\alpha)$ and the $\mathrm{O}(\beta)$ variations arise from the Chern–Simons forms in H, and these obviously have the same structure for the Yang-Mills and Lorentz groups.

One may wonder in what sense the R^2-invariant presented here is unique. Clearly, one could have considered terms containing the Ricci tensor $R_\mu{}^a(\Omega_-)\equiv R_{\mu\nu}{}^{a\nu}(\Omega_-)$, or the Ricci scalar $R(\Omega_-)\equiv R_\mu{}^\mu(\Omega_-)$, as well. However, the Ricci tensor is the leading term in the zehnbein equation of motion to order α^0. Therefore such contributions to an αR^2-action can always be cancelled by a redefinition of the zehnbein of order α. Thus there is an ambiguity in the supersymmetrization of Chern–Simons forms, corresponding to such redefinitions. In the following sections we will sometimes employ this possibility of redefining fields to simplify actions and transformation rules. It should be stressed however, that the terms containing the full Riemann tensor, such as the R^2-term in eq. (2.11), are not affected by such redefinitions, and uniquely characterize the invariant action to this order in the parameters α and β.

3. The O(α^2) and O($\alpha\beta$) terms in the effective action

It is not too difficult to extend the results of the previous section beyond O(α) and O(β). In ref. [25] we made the following claim: no $\alpha^2 R^3 + \alpha\beta R F^2$ action is needed to supersymmetrize the Lorentz and Yang–Mills Chern–Simons term to O(α^2) and O($\alpha\beta$), but new O(α^2) and O($\alpha\beta$) variations of the supergravity fields are generated. These additional O(α^2) and O($\alpha\beta$) variations of the supergravity fields have been given in ref. [25].

The new variations were presented in ref. [25] in a form which is rather inconvenient for the calculation of the quartic effective action, the subject of the next section. The new transformation rules are unconventional in the sense that they contain derivatives of the supersymmetry parameter. This is rather unpleasant since many technical tricks involving supercovariantizations which we would like to use during our calculation rely on the fact that such derivatives do not occur.

We will show that by using a slightly different cancellation mechanism than considered in ref. [25] the undesirable feature mentioned above can be circumvented. The only price we have to pay is that the action will contain a few explicit O(α^2) and O($\alpha\beta$) terms (see eq. (3.17)).

As was explained in the previous section the O(α^2) and O($\alpha\beta$) variations of the action are completely determined by substituting $\delta\Omega_{\mu-}{}^{ab}$ and $\delta\psi^{ab}$, given in eq. (2.14), in the action (2.6) and (2.11). We then use the following lemma:

Lemma. For arbitrary transformations $\delta\Omega_{\mu-}{}^{ab}$ and $\delta\psi^{ab}$ the variation of $\mathscr{L}(R)+\mathscr{L}(R^2)$, given in eqs. (2.6) and (2.11), is given by

$$\delta\mathscr{L}=\delta_1\mathscr{L}+\delta_2\mathscr{L}\,, \tag{3.1}$$

with

$$\begin{aligned}\delta_1\mathscr{L}=\alpha\delta\bar{\psi}^{\lambda\rho}\Big[&2e\phi^{-3}\mathscr{D}_\lambda(\Omega_+)\Big(e^{-1}\phi^3\Big(\Psi_\rho+\tfrac{1}{4}\sqrt{2}\,\Gamma_\rho\Lambda\Big)\Big)\\ &-\Gamma^a\psi_\lambda\Big(\mathscr{E}_{\rho a}+\tfrac{1}{3}e_{\rho a}\Phi-\sqrt{2}\,\mathscr{B}_{\rho a}\Big)\Big]+\alpha\sqrt{2}\,\mathscr{B}^{\mu\nu}\Omega_{\mu-}{}^{ab}\delta\Omega_{\nu-}{}^{ab}\\ &-2\alpha\Big(\mathscr{E}_\rho{}^a+\tfrac{1}{3}e_\rho{}^a\Phi-\sqrt{2}\,B_\rho{}^a\Big)\Big[e^{-1}\phi^3\mathscr{D}_\lambda(\Omega_+)\Big(e\phi^{-3}e^{\nu a}\delta\Omega_{\nu-}{}^{\lambda\rho}\Big)\Big]\,,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\delta_2\mathscr{L}=\;&3\alpha\mathscr{D}_\mu(\Omega_-)\Big(e\phi^{-3}T^{\mu\nu ab}\Big)\delta\Omega_{\nu-}{}^{ab}+\tfrac{9}{2}\alpha\sqrt{2}\,e\phi^{-3}H^{\nu\lambda\rho}T_{\lambda\rho ab}\delta\Omega_{\nu-}{}^{ab}\\ &-\tfrac{1}{2}\alpha e\phi^{-3}\delta\bar{\psi}^{\lambda\rho}\Gamma^{\mu ab}\psi_\lambda T_{\rho\mu ab}-\tfrac{3}{4}\alpha e\phi^{-3}\delta\bar{\psi}^{\lambda\rho}\Gamma^\mu\Gamma^{ab}\Big(\psi_\mu+\tfrac{1}{3}\sqrt{2}\,\Gamma_\mu\lambda\Big)T_{\lambda\rho ab}\,,\end{aligned} \tag{3.3}$$

where $T_{\mu\nu\rho\sigma}$ is defined in eq. (2.15) and $\mathscr{E}^\mu{}_a$, Φ, Ψ^μ, Λ and $\mathscr{B}^{\mu\nu}$ are the equations of motion to lowest order in α and β (i.e. following from (2.6)) of the supergravity fields $e_\mu{}^a$, ϕ, ψ_μ, λ and $B_{\mu\nu}$, respectively.

In the lemma the lowest order equations of motion ($O(\alpha^0)$) play a crucial role, and therefore it is useful to present these equations at this stage. The equations of motion for the supergravity fields are

$$\Phi = e\phi^{-3}\left(\tfrac{3}{2}R(\omega) - 9\mathscr{D}_a(\omega)(\phi^{-1}\partial_a\phi) + \tfrac{27}{2}(\phi^{-1}\partial_\mu\phi)^2 + \tfrac{9}{4}H^{\mu\nu\lambda}H_{\mu\nu\lambda}\right), \quad (3.4)$$

$$\mathscr{E}^\mu{}_a = e\phi^{-3}e^\mu{}_b e^{\nu a}\left(R_\nu{}^b(\omega) - 3\mathscr{D}_\nu(\omega)(\phi^{-1}\partial^b\phi) + \tfrac{9}{2}H_{\nu\lambda\rho}H^{b\lambda\rho}\right) - \tfrac{1}{3}e^\mu{}_a\Phi, \quad (3.5)$$

$$\Lambda = e\phi^{-3}\left(8\not{D}(\omega)\lambda + \sqrt{2}\,\Gamma^{\mu\nu}\psi_{\mu\nu} - 12(\phi^{-1}\not{D}\phi)\lambda - \sqrt{2}\,\Gamma^{abc}\lambda H_{abc}\right), \quad (3.6)$$

$$\Psi_\mu = e\phi^{-3}\left(\Gamma^\nu\psi_{\mu\nu} + 2\sqrt{2}\,D_\mu(\Omega_+)\lambda\right) - \tfrac{1}{4}\sqrt{2}\,\Gamma_\mu\Lambda, \quad (3.7)$$

$$\mathscr{B}^{\mu\nu} = \tfrac{3}{2}\partial_\lambda\left(e\phi^{-3}H^{\lambda\mu\nu}\right). \quad (3.8)$$

For the Yang–Mills fields we have

$$\mathscr{A}^\mu = e^\mu{}_a\mathscr{D}_\nu(\Omega_+, A)\left(e\phi^{-3}F^{\nu a}\right) - \tfrac{3}{2}\sqrt{2}\,A_\rho\,\partial_\lambda\left(e\phi^{-3}H^{\lambda\mu\rho}\right), \quad (3.9)$$

$$\mathscr{X} = -e\phi^{-3}\left[\not{D}(\omega, A)\chi + \tfrac{1}{2}\Gamma^{\mu\nu}F_{\mu\nu}\lambda - \tfrac{3}{2}(\phi^{-1}\not{\partial}\phi)\chi - \tfrac{1}{8}\sqrt{2}\,\Gamma^{\mu\nu\lambda}\chi H_{\mu\nu\lambda}\right]. \quad (3.10)$$

In the covariant derivatives in eqs. (3.9) and (3.10) the first covariantization concerns the Lorentz, the second the Yang–Mills structure. We present only the purely bosonic part of the bosonic equations of motion, since the terms bilinear in fermions play no role in eq. (3.2). Note that in this order $\mathscr{E}_{[\mu a]} = 0$.

The proof of the lemma proceeds as follows. An arbitrary variation of $\Omega_{\mu-}^{ab}$ and ψ^{ab} in $\mathscr{L}(R) + \mathscr{L}(R^2)$ gives

$$+\alpha\left[e^\mu{}_a\mathscr{D}_\nu(\Omega_+, \Omega_-)\left(e\phi^{-3}R^{\nu acd}(\Omega_-)\right) - \tfrac{3}{2}\sqrt{2}\,\Omega_{\rho-}^{cd}\,\partial_\lambda\left(e\phi^{-3}H^{\lambda\mu\rho}\right)\right]\delta\Omega_{\mu-}^{cd}$$

$$+ \alpha\delta\bar{\psi}^{ab}\Big[-e\phi^{-3}\Big\{\not{D}(\omega, \Omega_-)\psi^{ab} + \tfrac{1}{2}R_{\mu\nu}{}^{ab}(\Omega_-)\Gamma^{\mu\nu}\lambda$$

$$- \tfrac{3}{2}(\phi^{-1}\not{\partial}\phi)\psi^{ab} - \tfrac{1}{8}\sqrt{2}\,\Gamma^{\mu\nu\lambda}\psi^{ab}H_{\mu\nu\lambda}\Big\}\Big]. \quad (3.11)$$

The terms in brackets in eq. (3.11) are nothing but $\mathscr{A}^\mu$ and $\mathscr{X}$, but with the dependence on the Yang–Mills fields A_μ and χ everywhere replaced by $\Omega_{\mu-}^{ab}$ and ψ^{ab}. This can be understood from the complete symmetry between the R^2- and

F^2-actions. Eq. (3.11) can be simplified by using the identities

$$\mathscr{D}_\nu(\Omega_+,\Omega_-)\big(e\phi^{-3}R^{\nu acd}(\Omega_-)\big)$$
$$= 2e\phi^{-3}e^{\lambda c}e^{\rho d}\mathscr{D}_{[\lambda}(\Omega_+)\big(R_{\rho]}{}^a(\Omega_+) - 3\mathscr{D}_{\rho]}(\Omega_+)(\phi^{-1}\partial^a\phi)\big)$$
$$+ 3\mathscr{D}_\nu(\Omega_+,\Omega_-)\big(e\phi^{-3}T^{\nu acd}\big), \tag{3.12}$$

$$\not{D}(\omega,\Omega_-)\psi^{ab} + \tfrac{1}{2}R_{\mu\nu}{}^{ab}(\Omega_-)\Gamma^{\mu\nu}\lambda - \tfrac{3}{2}(\phi^{-1}\not\partial\phi)\psi^{ab} - \tfrac{1}{8}\sqrt{2}\,\Gamma^{\mu\nu\lambda}\psi^{ab}H_{\mu\nu\lambda}$$
$$= -2e^{\mu[a}e^{\nu b]}D_\mu(\Omega_+)\big(\Gamma^\rho\psi_{\nu\rho} + 2\sqrt{2}\,D_\nu(\Omega_+)\lambda\big)$$
$$+ \tfrac{1}{2}\Gamma_{\mu cd}\psi^{[a}T^{b]\mu cd} + \tfrac{3}{4}\Gamma^\mu\Gamma_{cd}\big(\psi_\mu + \tfrac{1}{3}\sqrt{2}\,\Gamma_\mu\lambda\big)T^{abcd}. \tag{3.13}$$

The derivation of these identities requires the use of the Bianchi identity for $R_{\mu\nu}{}^{ab}(\Omega_\pm)$ and ψ^{ab}. The additional terms containing $T^{\mu\nu ab}$ arise from the application of the Bianchi identity of $H_{\mu\nu\lambda}$. In the T term in eq. (3.12) the Ω_+ covariantization acts on the Lorentz index a, Ω_- on the indices cd. Finally, to obtain eqs. (3.2) and (3.3) one uses the identities

$$R_\mu{}^a(\Omega_+) - 3\mathscr{D}_\mu(\Omega_+)(\phi^{-1}\partial^a\phi) = e^{-1}\phi^3\big(\mathscr{E}_\mu{}^a + \tfrac{1}{3}e_\mu{}^a\Phi - \sqrt{2}\,\mathscr{B}_\mu{}^a\big), \tag{3.14}$$

$$\Gamma^\nu\psi_{\mu\nu} + 2\sqrt{2}\,D_\mu(\Omega_+)\lambda = e^{-1}\phi^3\big(\Psi_\mu + \tfrac{1}{4}\sqrt{2}\,\Gamma_\mu\Lambda\big). \tag{3.15}$$

Note that the combination of equations of motion which occur in eqs. (3.14) and (3.15) and in eq. (3.2) are precisely those in which the H-dependence can be absorbed in a spin connection with torsion Ω_+. Also, under supersymmetry these combinations transform into each other, e.g.,

$$\delta\big(\Psi_\mu + \tfrac{1}{4}\sqrt{2}\,\Gamma_\mu\Lambda\big) = \tfrac{1}{2}\Gamma^a\epsilon\big(\mathscr{E}_\mu{}^a + \tfrac{1}{3}e_\mu{}^a\Phi - \sqrt{2}\,\mathscr{B}_\mu{}^a\big) + \tfrac{1}{4}e\phi^{-3}\Gamma^{abc}\epsilon T_{\mu abc}. \tag{3.16}$$

This variation is important in the cancellation of $\delta_1\mathscr{L}$ in eq. (3.2). The term containing $T_{\mu\nu\lambda\rho}$ in eq. (3.16) is of higher order in α and β.

We now consider the above lemma for $\delta\psi^{ab}$ and $\delta\Omega_{\mu-}{}^{ab}$ as given in eq. (2.14). The terms given $\delta_1\mathscr{L}$ (see eq. (3.2)) determine the full $O(\alpha^2)$ and $O(\alpha\beta)$ variation of the $\mathscr{L}(R)+\mathscr{L}(R^2)+\mathscr{L}(F^2)$ action. Since all terms in (3.2) contain equations of motion of the supergravity fields it is in principle possible to cancel them all by modifying the supersymmetry transformation rules of the supergravity fields. This cancellation mechanism was used in ref. [25] but leads to the undesirable feature discussed in the beginning of this section. To avoid this we isolate the $\mathscr{D}\epsilon$ terms in eq. (3.2), which occur in the first (after a partial integration) and in the last term. These $\mathscr{D}\epsilon$ terms will be cancelled by adding new terms to the action which contain an explicit gravitino field. These new terms take the form

$$\mathscr{L}(R^3+RF^2) = -3\alpha\sqrt{2}\,\mathscr{B}^{\mu\nu}\bar{\psi}^{\lambda}\Gamma_{[\lambda}X_{\mu\nu]} - \tfrac{3}{2}\alpha\bar{\psi}_{\lambda}\Gamma_{ab}\left(\Psi_{\rho}+\tfrac{1}{4}\sqrt{2}\,\Gamma_{\rho}\Lambda\right)T^{\lambda\rho ab}. \quad (3.17)$$

Because in the present case $\delta\Omega_{\mu-}{}^{ab}$ (see eq. (2.14)) is completely antisymmetric, the zehnbein and ϕ equations of motion do not appear in eq. (3.17). The new terms in eq. (3.17) lead to further variations which have to be cancelled. They come from varying the fermionic field equation in the second term, and $X_{\mu\nu}$ in the first one. The variations are

$$-3\alpha\sqrt{2}\,\mathscr{B}^{\mu\nu}\bar{\psi}^{\lambda}\Gamma_{[\lambda}\delta X_{\mu\nu]} - \tfrac{3}{4}\alpha\bar{\psi}_{\lambda}\Gamma_{ab}\Gamma_{c}\epsilon T^{\lambda\rho ab}\left(\mathscr{E}_{\rho}{}^{c}+\tfrac{1}{3}e_{\rho}{}^{c}\Phi-\sqrt{2}\,\mathscr{B}_{\rho}{}^{c}\right)$$
$$-\tfrac{3}{8}\alpha e\phi^{-3}\bar{\psi}_{\lambda}\Gamma_{ab}\Gamma^{cde}\epsilon T^{\lambda\rho ab}T_{\rho cde}. \quad (3.18)$$

The last term in eq. (3.18) is of higher order in α and β, and must be taken into account in the next section. The other terms, and the remainder of eq. (3.2), we cancel by modifications of the supergravity transformation rules.

These modifications can then be written as follows

$$(\delta_{\alpha^2}+\delta_{\alpha\beta})\,e_{\mu}{}^{a} = -3\alpha\sqrt{2}\,H_{\mu\rho\nu}(\delta_{\alpha}+\delta_{\beta})\,\Omega_{-}^{a\rho\nu},$$

$$(\delta_{\alpha^2}+\delta_{\alpha\beta})\,\psi_{\mu} = \tfrac{3}{2}\alpha e^{-1}\phi^{3}\Gamma_{ab}\epsilon g_{\mu\rho}\mathscr{D}_{\lambda}(\Omega_{+})\left(e\phi^{-3}T^{\lambda\rho ab}\right),$$

$$(\delta_{\alpha^2}+\delta_{\alpha\beta})\,B_{\mu\nu} = -3\alpha\sqrt{2}\,g_{\rho[\mu}e_{\nu]}{}^{a}e^{-1}\phi^{3}\bar{\epsilon}D_{\lambda}(\Omega_{+})\left(e\phi^{-3}\Gamma^{[a}X^{\lambda\rho]}\right) - \alpha\sqrt{2}\,\Omega_{[\mu-}{}^{ab}(\delta_{\alpha}+\delta_{\beta})\,\Omega_{\nu]-}{}^{ab},$$

$$(\delta_{\alpha^2}+\delta_{\alpha\beta})\,\lambda = -\tfrac{1}{4}\sqrt{2}\,\Gamma^{\mu}(\delta_{\alpha^2}+\delta_{\alpha\beta})\,\psi_{\mu},$$

$$(\delta_{\alpha^2}+\delta_{\alpha\beta})\,\phi = 3e^{\mu}{}_{a}\phi(\delta_{\alpha^2}+\delta_{\alpha\beta})\,e_{\mu}{}^{a}. \quad (3.19)$$

We have thus established the main result of this section: an $O(\alpha^2)$, $O(\alpha\beta)$ supersymmetric extension of the Lorentz and Yang–Mills Chern–Simons term can be obtained by adding to the action given in eqs. (2.6), (2.7) and (2.11) the terms given in eq. (3.17) and by adding the new variations (3.19) to the transformation rules of the supergravity fields given in eqs. (2.1), (2.2) and (2.12). We note that the new terms (3.17) added to the action do not contain a bosonic R^3-term. This result agrees with superstring amplitude calculations [4–8] and superspace arguments [13–16], which also shows that a bosonic term must be absent at this level. The variation of $e_{\mu}{}^{a}$ in (3.19) is only determined up to a field-dependent local Lorentz transformation. On the other fields this local Lorentz transformation is either absent or of higher order in the fermions, and can therefore be ignored.

Of course the result obtained in this section is not supersymmetric to higher order in α, β, i.e. $O(\alpha^3)$, $O(\alpha^2\beta)$ and $O(\alpha\beta^2)$. To obtain an invariant up to that order new terms have to be added to the action which are of the form α^3R^4, $\alpha^2\beta F^2R^2$ and $\alpha\beta^2F^4$. The determination of the structure of these terms will be the subject of the next section. In this section we have already encountered a number of terms which will play a role in the determination of the $O(\alpha^3, \alpha^2\beta, \alpha\beta^2)$ action. Since there are several distinct sources we will discuss them in order.

(i) The new variations of the supergravity fields lead to $O(\alpha^2)$ and $O(\alpha\beta)$ variations of $\Omega_{\mu-}^{\ ab}$ and ψ^{ab}. The substitution of these new transformation in the $\mathscr{L}(R)+\mathscr{L}(R^2)$ action gives variations of the desired order. Clearly, all the new variations from this source can be cancelled by applying the lemma of this section. The contributions from this source to the action and the transformation rules will be further discussed in sect. 4.

The full transformation rule of ψ_μ, given in eqs. (2.1) and (3.19), can now be written as

$$\delta\psi_\mu = \mathscr{D}_\mu(\hat{\Omega}_+)\epsilon\,, \tag{3.20}$$

where

$$\hat{\Omega}_{\mu\pm}^{\ ab} = \Omega_{\mu\pm}^{\ ab} \mp 6\alpha e^{-1}\phi^3 g_{\mu\rho}\mathscr{D}_\lambda(\Omega_+)\left(e\phi^{-3}T^{\lambda\rho ab}\right). \tag{3.21}$$

This suggests that it might be advantageous to modify the definition of $\Omega_\pm$ from order to order. However, since we do not consider the effective action beyond $O(\alpha^3, \alpha^2\beta, \alpha\beta^2)$, the advantage of such a redefinition in the present calculation is slight. Although the modification (3.21) helps to preserve the symmetry between supergravity and Yang–Mills multiplets, one should note that the transformations (2.14) have already broken this symmetry.

(ii) The new transformations of the supergravity fields themselves have also to be applied in the variation of the $\alpha R^2+\beta F^2$-action. In fact, since we neglect higher order fermionic terms, we only have to vary ψ_μ and λ in the Noether terms, and the zehnbein and ϕ in the purely bosonic terms.

(iii) In the calculation of this section we have used a number of times the Bianchi identity for the tensor $H_{\mu\nu\lambda}$. This identity leads to T-terms (see eqs. (3.3) and (3.16)), which have now to be taken into account.

In summary, we have three sources for the $O(\alpha^3)$, $O(\alpha^2\beta)$ and $O(\alpha\beta^2)$ variations which we have specified above. The ones coming from (i) can be cancelled by using the lemma. The ones coming from (ii) and (iii) are given by

$$\begin{aligned}
&\tfrac{9}{2}\alpha\mathscr{D}_\mu(\Omega_-)\left(e\phi^{-3}T^{\mu\nu ab}\right)\bar{\epsilon}\Gamma_\nu X_{ab} \\
&\quad + \tfrac{27}{4}\alpha\sqrt{2}\,e\phi^{-3}H^{\nu\lambda\rho}T_{\lambda\rho ab}\bar{\epsilon}\Gamma_{[\nu}X_{ab]} + \tfrac{3}{8}\alpha e\phi^{-3}\bar{\epsilon}\Gamma^{cd}\Gamma^{\mu ab}\psi_\lambda T_{\rho\mu ab}T^{\lambda\rho cd} \\
&\quad + \tfrac{9}{16}\alpha e\phi^{-3}\bar{\epsilon}\Gamma^{cd}\Gamma^{\mu}\Gamma^{ab}\left(\psi_\mu + \tfrac{1}{3}\sqrt{2}\,\Gamma_\mu\lambda\right)T^{\lambda\rho cd}T_{\lambda\rho ab} \\
&\quad + \tfrac{3}{8}\alpha e\phi^{-3}\bar{\epsilon}\Gamma^{\mu ab}\Gamma^{cd}\psi_\lambda T_{\rho\mu ab}T^{\lambda\rho cd} + \tfrac{3}{2}\alpha\bar{\epsilon}\Gamma_{ab}\Gamma^{\mu}X_{\mu\rho}\mathscr{D}_\lambda(\Omega_+)\left(e\phi^{-3}T^{\lambda\rho ab}\right) \\
&\quad + \tfrac{9}{2}\alpha\sqrt{2}\,e\phi^{-3}\bar{\epsilon}\Gamma_{[a}X_{bc]}H^{\mu bc}T_\mu^{\ a}\,,
\end{aligned} \tag{3.22}$$

where $T_{\mu\nu} = \alpha R_{\mu\lambda}{}^{ab} R^{\lambda}{}_{\nu}{}^{ab} + \beta \operatorname{tr} F_{\mu\lambda} F^{\lambda}{}_{\nu}$. Note that only the last two terms come from (ii). The terms given in eq. (3.22) are the ones which have to be cancelled by the addition of new terms to the action.

Before ending this section, we would like to comment on the variations given in eq. (3.22). Quite surprisingly, we see that all terms contain the Riemann curvature tensor $R_{\mu\nu}{}^{ab}$ and the Yang–Mills curvature $F_{\mu\nu}$ only through the tensor T_{abcd} and the tensor-spinor X_{ab} which are defined in eq. (2.15). This means that in the variations there is a complete symmetry between (R^{ab}, ψ^{ab}) and (F, χ):

$$R_{\mu\nu}{}^{ab} \leftrightarrow F_{\mu\nu}\,, \qquad \psi^{ab} \leftrightarrow \chi\,. \tag{3.23}$$

We therefore expect that the new terms to be added to the action in the next order will also exhibit this symmetry and can be formulated completely in terms of the T and X tensors. This agrees with the results found from string amplitude calculations. This observation will simplify enormously the ansatz which we will make in the next section for the quartic effective action.

4. The $O(\alpha^3)$, $O(\alpha^2\beta)$ and $O(\alpha\beta^2)$ terms in the effective action

In the previous section we have already encountered a number of contributions to the variation of the action which were of $O(\alpha^3, \alpha^2\beta, \alpha\beta^2)$. These terms were collected in eq. (3.22), and cannot be trivially cancelled by a modification of the transformation rules. Therefore we expect the presence of terms containing $\alpha^3 R^4$, $\alpha^2\beta R^2 F^2$ and $\alpha\beta^2 F^4$, as well as their fermionic counterparts, in the action. This section will be devoted to their construction. We will present an ansatz for the action, and show that the requirement of supersymmetry and the ansatz are consistent. Before we present this ansatz, it will be useful to discuss the ingredients that go into it.

First of all, following the philosophy of the previous sections, we will use everywhere the Riemann tensor in the form $R_{\mu\nu}{}^{ab}(\Omega_-)$. As we have seen, it is in this form that the symmetry between the Lorentz and Yang–Mills parts of the action is manifest. Secondly, we will not include any terms in the action which depend on the Ricci tensor or other lowest order equations of motion, and therefore vanish on-shell. Such terms can always be cancelled by a modification of supergravity or Yang–Mills fields, and therefore play no role in the cancellation of the terms given in eq. (3.22).

In eq. (3.22) we have seen that the contributions to the Yang–Mills and Lorentz sectors occur in the specific combinations $T_{\mu\nu\lambda\rho}$, $T_{\mu\nu}$, T and $X_{\mu\nu}$. For completeness and clarity let us gather some definitions at this point:

$$T_{\mu\nu\lambda\rho} \equiv \alpha R_{[\mu\nu}{}^{ab}(\Omega_-) R_{\lambda\rho]}{}^{ab}(\Omega_-) + \beta \operatorname{tr} F_{[\mu\nu} F_{\lambda\rho]} ,$$

$$T_{\mu\nu} \equiv \alpha R_{\mu\lambda}{}^{ab}(\Omega_-) R^{\lambda}{}_{\nu}{}^{ab}(\Omega_-) + \beta \operatorname{tr} F_{\mu\lambda} F^{\lambda}{}_{\nu} , \qquad T \equiv T_{\mu}{}^{\mu} , \tag{4.1}$$

$$X_{\mu\nu} \equiv \alpha R_{\mu\nu}{}^{ab}(\Omega_-) \psi^{ab} + \beta \operatorname{tr} F_{\mu\nu} \chi ,$$

$$X_{\mu} \equiv \Gamma^{\nu} X_{\mu\nu} , \qquad X \equiv \Gamma^{\mu} X_{\mu} . \tag{4.2}$$

We will assume that also in the $O(\alpha^3, \alpha^2\beta, \alpha\beta^2)$ action the contributions from the Lorentz and Yang–Mills sector will occur in the combinations (4.1) and (4.2) (except for a contribution to the fermionic sector, as we will see in eq. (4.6)). This assumption contains two important ingredients. The first is that it enforces the symmetry between the Lorentz and Yang–Mills sectors also in the next order in α and β. This part of the assumption is consistent with eq. (3.22), and also agrees with the result of string amplitude calculations [5]. The second ingredient is that no combinations of curvatures appear in the action other than those given in eq. (4.1). One combination which is also symmetric in the Lorentz and Yang–Mills parts is missing from eq. (4.1); it is

$$U_{\mu\nu\lambda\rho} \equiv \tfrac{2}{3}\alpha R_{\mu(\nu}{}^{ab} R_{\lambda)\rho}{}^{ab} + \tfrac{2}{3}\beta \operatorname{tr} F_{\mu(\nu} F_{\lambda)\rho} - [\mu \leftrightarrow \nu] \tag{4.3}$$

(the representation of dimension 770 of SO(9, 1)). By not including terms depending on eq. (4.3) in the bosonic part of the action we follow again the result of string calculations. As is well known, the terms quartic in R and F in the action should have the generic structure

$$t^{\mu\nu\lambda\rho\alpha\beta\sigma\tau}\Big(R_{\mu\nu}{}^{ab} R_{\lambda\rho}{}^{ab} + \operatorname{tr} F_{\mu\nu} F_{\lambda\rho} \Big)\Big(R_{\alpha\beta}{}^{cd} R_{\sigma\tau}{}^{cd} + \operatorname{tr} F_{\alpha\beta} F_{\sigma\tau} \Big) , \tag{4.4}$$

where the tensor t is given in, e.g., ref. [29]. On working out (4.4), using the explicit form of the t-tensor, one finds that (4.4) contains only the combinations T given in eq. (4.1), and not U (4.3). Therefore we will start with an action that does not contain (4.3), and the requirement of supersymmetry will then decide on the validity of this assumption.

That this assumption is not entirely innocent follows from a look at the supersymmetry transformations of the tensors defined in (4.2). These are

$$\delta X_{\mu\nu} = -\tfrac{1}{4}\Gamma^{\lambda\rho}\epsilon\big(T_{\mu\nu\lambda\rho} + U_{\mu\nu\lambda\rho}\big) , \qquad \delta X_{\mu} = -\tfrac{1}{4}\Gamma^{\nu\lambda\rho}\epsilon T_{\mu\nu\lambda\rho} - \tfrac{1}{2}\Gamma^{\rho}\epsilon T_{\mu\rho} ,$$

$$\delta X = -\tfrac{1}{4}\Gamma^{\mu\nu\lambda\rho}\epsilon T_{\mu\nu\lambda\rho} - \tfrac{1}{2}\epsilon T . \tag{4.5}$$

Therefore, any dependence of the action on $X_{\mu\nu}$ will potentially produce U-terms in the variation.

We emphasize that there exists another part of the R^4-action, which is not related to the supersymmetrization of the Chern–Simons forms, and has a structure containing two t-tensors [4]. This term does not have a corresponding Yang–Mills counterpart. We will briefly discuss this term in the conclusions.

We therefore consider the most general form of the $O(\alpha^3, \alpha^2\beta, \alpha\beta^2)$ action required for the supersymmetrization of the Chern–Simons forms:

$$
\begin{aligned}
&\alpha^{-1}e^{-1}\phi^3\mathscr{L}(R^4+R^2F^2+F^4)\\
&\quad = +a_1 T^{\mu\nu\lambda\rho}T_{\mu\nu\lambda\rho} + a_2 T^{\mu\nu}T_{\mu\nu} + a_3 T^2 + b_1 T^{\mu\nu}\left(\alpha\bar{\psi}^{ab}\Gamma_\mu\,\partial_\nu\psi^{ab} + \beta\,\mathrm{tr}\,\bar{\chi}\Gamma_\mu\,\partial_\nu\chi\right)\\
&\qquad + b_2\partial_\mu T^{\mu\nu\lambda\rho}\left(\alpha\bar{\psi}^{ab}\Gamma_{\nu\lambda\rho}\psi^{ab} + \beta\,\mathrm{tr}\,\bar{\chi}\Gamma_{\nu\lambda\rho}\chi\right)\\
&\qquad + b_3\bar{X}_\mu\,\partial_\nu X^{\mu\nu} + b_4\bar{X}_{\mu\nu}\,\not{\partial}X^{\mu\nu} + b_5\bar{X}_\mu\,\not{\partial}X^\mu + b_6\bar{X}\not{\partial}X\\
&\qquad + T^{\mu\nu\lambda\rho}\big(c_1\bar{\psi}_\mu\Gamma_\nu X_{\lambda\rho} + c_2\bar{\psi}_\sigma\Gamma^\sigma{}_{\mu\nu}X_{\lambda\rho} + c_3\bar{\psi}_\sigma\Gamma_{\mu\nu\lambda}X^\sigma{}_\rho\\
&\qquad\quad + c_4\psi_\mu\Gamma_{\nu\lambda}X_\rho + c_5\bar{\psi}_\sigma\Gamma^\sigma{}_{\mu\nu\lambda}X_\rho + c_6\bar{\psi}_\sigma\Gamma_{\mu\nu\lambda\rho}X^\sigma + c_7\bar{\psi}_\mu\Gamma_{\nu\lambda\rho}X + c_8\bar{\psi}_\sigma\Gamma^\sigma{}_{\mu\nu\lambda\rho}X\big)\\
&\qquad + T^{\mu\nu}\left(d_1\bar{\psi}_\sigma\Gamma_\mu X^\sigma{}_\nu + d_2\bar{\psi}_\mu X_\nu + d_3\bar{\psi}_\sigma\Gamma^\sigma{}_\mu X_\nu + d_4\bar{\psi}_\mu\Gamma_\nu X\right)\\
&\qquad + T\left(e_1\bar{\psi}_\sigma X^\sigma + e_2\bar{\psi}_\sigma\Gamma^\sigma X\right) + T^{\mu\nu\lambda\rho}\left(f_1\bar{\lambda}\Gamma_{\mu\nu}X_{\lambda\rho} + f_2\bar{\lambda}\Gamma_{\mu\nu\lambda}X_\rho + f_3\bar{\lambda}\Gamma_{\mu\nu\lambda\rho}X\right)\\
&\qquad + T^{\mu\nu}f_4\bar{\lambda}\Gamma_\mu X_\nu + Tf_5\bar{\lambda}X.
\end{aligned}
\tag{4.6}
$$

In the fermionic terms (b_1 and b_2) we see that combinations of fields other than eqs. (4.1) and (4.2) appear. We have not included terms containing explicit $\phi^{-1}\partial\phi$ or H-contributions. These will be discussed later. The determination of the coefficients in eq. (4.6) is independent of such additional terms.

For the cancellation of eq. (3.22) we require the $O(\alpha^3, \alpha^2\beta, \alpha\beta^2)$ variation of eq. (4.6). Therefore we have to use the $O(\alpha^0, \beta^0)$ variation of the supergravity and Yang–Mills fields in eq. (4.6), and the variation (4.5) for $X_{\mu\nu}$, X_μ and X. The variation of $T_{\mu\nu\lambda\rho}$, $T_{\mu\nu}$, and T cannot be expressed directly in terms of $X_{\mu\nu}$, X_μ and X. Instead we have, e.g.

$$
\delta T_{\mu\nu\lambda\rho} = \alpha R_{[\mu\nu}{}^{ab}(\Omega_-)\,\partial_\lambda\left(\bar{\epsilon}\Gamma_{\rho]}\psi^{ab}\right) + \beta\,\mathrm{tr}\,F_{[\mu\nu}\partial_\lambda\left(\bar{\epsilon}\Gamma_{\rho]}\chi\right). \tag{4.7}
$$

This, and the terms with b_1 and b_2 in eqs. (4.6), are the only sources that produce explicit $F_{\mu\nu}$ and χ in the variation of eq. (4.6). If such terms cannot be rewritten in terms of X or T they must cancel amongst each other *or* give terms proportional to equations of motion.

It is worthwhile to go into some detail concerning these equation of motion terms. The lowest order equations of motion for the Yang–Mills fields were presented in (3.9) and (3.10). In the variation of eq. (4.6) we sill encounter contributions proportional to these equations of motion. They can be cancelled by appropriate modifications to the supersymmetry transformation rules of the Yang–Mills fields A_μ and χ. For every equation of motion of a Yang–Mills field, there will be a corresponding contribution to the Lorentz sector. Since $\Omega_{\mu-}^{ab}$ and ψ^{ab} are not independent fields, these terms cannot be immediately interpreted as equations of motion. However, as we have seen in eqs. (3.11)–(3.15), all such terms can be rewritten in terms of equations of motion of the supergravity fields, and can therefore be cancelled by changing the transformation rules of the supergravity fields.

The coefficients in (4.6) can be uniquely determined by considering variations of the generic form $\epsilon\lambda T^2$, $\partial T\epsilon X$ and $T\epsilon\partial X$. The results is as follows:

$$\begin{array}{lllll}
a_1 = \frac{3}{2}\,, & a_2 = \frac{1}{2}\,, & a_3 = 0\,, & & \\
b_1 = -\frac{1}{2}\,, & b_2 = -\frac{1}{8}\,, & b_3 = 0\,, & & \\
b_4 = 0\,, & b_5 = -\frac{1}{2}\,, & b_6 = -\frac{1}{16}\,, & & \\
c_1 = 0\,, & c_2 = 0\,, & c_3 = 0\,, & c_4 = -\frac{3}{4}\,, & \\
c_5 = \frac{1}{4}\,, & c_6 = \frac{1}{16}\,, & c_7 = -\frac{1}{16}\,, & c_8 = \frac{1}{64}\,, & \\
d_1 = 1\,, & d_2 = 1\,, & d_3 = 0\,, & d_4 = -\frac{1}{4}\,, & \\
e_1 = \frac{1}{8}\,, & e_2 = \frac{1}{32}\,, & & & \\
f_1 = 0\,, & f_2 = -\frac{1}{2}\sqrt{2}\,, & f_3 = -\frac{1}{32}\sqrt{2}\,, & f_4 = 0\,, & f_5 = -\frac{1}{16}\sqrt{2}\,.
\end{array} \tag{4.8}$$

We see that most of the terms containing $X_{\mu\nu}$ (the exception being d_1) vanish. This is not surprising, since $X_{\mu\nu}$ is the only source of the tensor $U_{\mu\nu\lambda\rho}$ (4.3). In determining the coefficients, it is essential to use the following differential relations:

$$\partial_\nu T^{\mu\nu} = \tfrac{1}{4}\,\partial^\mu T + \alpha\big(\partial_\lambda R_\nu{}^{\lambda ab}(\Omega_-)\big) R^{\mu\nu ab}(\Omega_-) + \beta\,\mathrm{tr}\big(\partial_\lambda F_\nu{}^\lambda\big) F^{\mu\nu}\,, \tag{4.9}$$

$$\begin{aligned}
\partial_\mu X^\mu = {}& \tfrac{1}{4}\,\not{\partial} X - \tfrac{1}{4}\Gamma^{\mu\nu}\big(\alpha R_{\mu\nu}{}^{ab}(\Omega_-)\,\not{\partial}\psi^{ab} + \beta\,\mathrm{tr}\,F_{\mu\nu}\,\not{\partial}\chi\big)\\
& - \tfrac{1}{2}\Gamma^\mu\Big(\alpha\big(\partial_\lambda R_\mu{}^{\lambda ab}(\Omega_-)\big)\psi^{ab} + \beta\,\mathrm{tr}\big(\partial_\lambda F_\mu^{\lambda}\big)\chi\Big)\,,
\end{aligned} \tag{4.10}$$

$$\partial_{[\mu} T_{\nu\lambda\rho\sigma]} = 0\,. \tag{4.11}$$

The relations (4.9) and (4.10) are among the many sources of equation of motion terms in the variation of eq. (4.6). In isolating the equation of motion terms one must keep in mind that the fermionic field equation must be supercovariant, and also contains a λ-dependent term (e.g. eq. (3.10)).

There is an essential check on the above calculation, which is the cancellation of all variations of (4.6) of the form $\epsilon\psi_\mu T^2$ against the corresponding terms in (3.22). We have verified that this cancellation takes place.

In the calculation leading to (4.8) we are left with a number of terms which are proportional to equations of motion. These must be cancelled in the way discussed above (4.8). Explicitly, these terms are for the Yang–Mills sector:

$$\begin{aligned} &e\phi^{-3}\alpha\beta\left(\partial_\lambda F_\nu{}^\lambda\right)\left[\tfrac{1}{2}F^{\mu\nu}\bar{\epsilon}X_\mu + \tfrac{1}{8}F^{\mu\nu}\bar{\epsilon}\Gamma_\mu X\right. \\ &\qquad\left. -\tfrac{1}{2}T^{\mu\nu}\bar{\epsilon}\Gamma_\mu\chi + \tfrac{1}{16}T\bar{\epsilon}\Gamma^\nu\chi + \tfrac{1}{32}T^{\alpha\beta\gamma\delta}\bar{\epsilon}\Gamma_{\alpha\beta\gamma\delta}\Gamma^\nu\chi\right] \\ &+ e\phi^{-3}\alpha\beta\left[\tfrac{1}{2}T^{\mu\nu}\bar{\epsilon}\Gamma_\mu\Gamma^\lambda F_{\nu\lambda} + \tfrac{1}{32}T\bar{\epsilon}\Gamma^{\mu\nu}F_{\mu\nu}\right. \\ &\qquad\left. + \tfrac{1}{64}T^{\alpha\beta\gamma\delta}\bar{\epsilon}\Gamma_{\alpha\beta\gamma\delta}\Gamma^{\mu\nu}F_{\mu\nu}\right]\left(\not{D}(\omega)\chi + \tfrac{1}{2}\Gamma^{\lambda\rho}F_{\lambda\rho}\lambda\right). \end{aligned} \tag{4.12}$$

As we can see in eq. (4.12), the calculation produces only the leading terms of the equations of motion (3.9) and (3.10). These leading terms should be completed to the full equations of motion. However, the missing terms all are of higher order in the fermions, or contain explicit $\phi^{-1}\partial\phi$ or H-contributions. These last terms will be discussed later.

The cancellation of eq. (4.12) is achieved by additional supersymmetry transformations of A_μ and χ:

$$\begin{aligned} \left(\delta_{\alpha^2\beta} + \delta_{\alpha\beta^2}\right)A^\nu = \alpha\beta&\left[\tfrac{1}{2}F^{\mu\nu}\bar{\epsilon}X_\mu + \tfrac{1}{8}F^{\mu\nu}\bar{\epsilon}\Gamma_\mu X\right. \\ &\left.-\tfrac{1}{2}T^{\mu\nu}\bar{\epsilon}\Gamma_\mu\chi + \tfrac{1}{16}T\bar{\epsilon}\Gamma^\nu\chi + \tfrac{1}{32}T^{\alpha\beta\gamma\delta}\bar{\epsilon}\Gamma_{\alpha\beta\gamma\delta}\Gamma^\nu\chi\right], \end{aligned} \tag{4.13}$$

$$\left(\delta_{\alpha^2\beta} + \delta_{\alpha\beta^2}\right)\chi = \alpha\beta\left[\tfrac{1}{2}T^{\mu\nu}\Gamma^\lambda\Gamma_\mu\epsilon F_{\nu\lambda} - \tfrac{1}{32}\Gamma^{\mu\nu}\epsilon F_{\mu\nu}T - \tfrac{1}{64}\Gamma^{\mu\nu}\Gamma_{\alpha\beta\gamma\delta}\epsilon F_{\mu\nu}T^{\alpha\beta\gamma\delta}\right]. \tag{4.14}$$

An important consistency check follows from the contribution of these terms to the commutator algebra on A_μ. These new terms give rise to terms of the form $\alpha\beta\bar{\epsilon}_2\Gamma_{\dots}\epsilon_1 F_{..}T_{\dots}$ in the commutator algebra. Since such terms cannot be interpreted as a gauge transformation of A_μ they should cancel, as indeed they do.

Terms similar to eq. (4.12), but with A_μ and χ replaced by $\Omega_{\mu-}{}^{ab}$ and ψ^{ab}, appear for the Lorentz sector. As explained above, these terms are cancelled by new variations of the supergravity fields. To achieve this cancellation, we use (3.12)–(3.15) to rewrite these terms in the variation of the action in the following form:

$$\alpha^2 \bar{\epsilon} N^{\lambda\rho}\Big[2e\phi^{-3}\mathscr{D}_\lambda(\Omega_+)\Big(e^{-1}\phi^3\Big(\Psi_\rho + \tfrac{1}{4}\sqrt{2}\,\Gamma_\rho\Lambda\Big)\Big)$$

$$-\Gamma^a\psi_\lambda\Big(\mathscr{E}_\rho^{\ a} + \tfrac{1}{3}e_\rho^{\ a}\Phi - \sqrt{2}\,\mathscr{B}_\rho^{\ a}\Big)\Big] + \alpha^2\sqrt{2}\,\mathscr{B}^{\mu\nu}\Omega_{\mu -}^{\ ab}\bar{\epsilon}M_\nu^{\ ab}$$

$$-2\alpha^2\Big(\mathscr{E}_\rho^{\ a} + \tfrac{1}{3}e_\rho^{\ a}\Phi - \sqrt{2}\,\mathscr{B}_\rho^{\ a}\Big)\Big[e^{-1}\phi^3\mathscr{D}_\lambda(\Omega_+)\Big(e\phi^{-3}e^{\nu a}\bar{\epsilon}M_\nu^{\ \lambda\rho}\Big)\Big]\,, \tag{4.15}$$

where

$$M_\mu^{\ ab} = \tfrac{1}{2}R_{\lambda\mu}^{\ \ ab}(\Omega_-)X^\lambda + \tfrac{1}{8}R_{\lambda\mu}^{\ \ ab}(\Omega_-)\Gamma^\lambda X$$

$$-\tfrac{1}{2}T_{\lambda\mu}\Gamma^\lambda\psi^{ab} + \tfrac{1}{16}T\Gamma_\mu\psi^{ab} + \tfrac{1}{32}T^{\alpha\beta\gamma\delta}\Gamma_{\alpha\beta\gamma\delta}\Gamma_\mu\psi^{ab}\,, \tag{4.16}$$

$$\bar{\epsilon}N^{\lambda\rho} = -\bar{\epsilon}\Big\{\tfrac{1}{32}T\Gamma^{\mu\nu}R_{\mu\nu}^{\ \ \lambda\rho}(\Omega_-) + \tfrac{1}{2}T^{\mu\nu}\Gamma_\mu\Gamma^\sigma R_{\nu\sigma}^{\ \ \lambda\rho}(\Omega_-)$$

$$+\tfrac{1}{64}T^{\alpha\beta\gamma\delta}\Gamma_{\alpha\beta\gamma\delta}\Gamma^{\mu\nu}R_{\mu\nu}^{\ \ \lambda\rho}(\Omega_-)\Big\}\,. \tag{4.17}$$

A comparison shows that eqs. (4.15) and (3.2) have exactly the same form, with $\delta\psi^{\lambda\rho}$ replaced by $\alpha\bar{\epsilon}N^{\lambda\rho}$, and $\delta\Omega_{\nu -}^{\ ab}$ by $\alpha\bar{\epsilon}M_\nu^{\ ab}$. The reason for this is as follows. The terms (4.12), with the equations of motion of the Yang–Mills sector, correspond to a variation of the $R+\beta F^2$ action with respect to the Yang–Mills fields A_ν and χ, multiplied by a variation of A_ν and χ. Because of the complete symmetry between the Lorentz and Yang–Mills parts in the variation of eq. (4.6) and in the $O(\alpha,\beta)$ action, the terms analogous to eq. (4.12) for the Lorentz sector can be written as variations of the $R+\alpha R^2$ action with respect to $\Omega_{\mu -}^{\ ab}$ and ψ^{ab}. Therefore these terms must take on the form of the lemma in sect. 3, with the variations $\delta\Omega$ and $\delta\psi$ replaced by eqs. (4.16) and (4.17) respectively.

It is now a simple matter to obtain from eq. (4.15) the new transformation rules of the supergravity fields. As in sect. 3, we prefer to avoid $\mathscr{D}\epsilon$ terms in the transformation rules. Therefore we isolate in eq. (4.15) the $\mathscr{D}\epsilon$ terms, and cancel these by adding new terms to the action. These new terms are:

$$+2\alpha^2\bar{\psi}_\lambda N^{\lambda\rho}\Big(\Psi_\rho + \tfrac{1}{4}\sqrt{2}\,\Gamma_\rho\Lambda\Big) - 2\alpha^2\Big(\mathscr{E}_\rho^{\ a} + \tfrac{1}{3}e_\rho^{\ a}\Phi - \sqrt{2}\,\mathscr{B}_\rho^{\ a}\Big)\bar{\psi}_\lambda e^{\nu a}M_\nu^{\ \lambda\rho}. \tag{4.18}$$

These terms also give new variations: in the first term we must vary the fermionic equation of motion, in the second $M_\nu^{\ \lambda\rho}$. The new variations combine with eq. (4.15), and the remainder is finally cancelled by the following additional transformations of the supergravity fields:

$$\left(\delta_{\alpha^3}+\delta_{\alpha^2\beta}\right)e_\mu{}^a = -2\alpha^2 e^{-1}\phi^3 e_\rho{}^a e_\mu{}^b \bar\epsilon D_\lambda(\Omega_+)\left(e\phi^{-3}e^{\nu b}M_\nu{}^{\lambda\rho}\right),$$

$$\left(\delta_{\alpha^3}+\delta_{\alpha^2\beta}\right)B_{\mu\nu} = -2\alpha^2\sqrt{2}\,g_{\rho[\mu}e_{\nu]}{}^a e^{-1}\phi^3\bar\epsilon D_\lambda(\Omega_+)\left(e\phi^{-3}e^{\sigma a}M_\sigma{}^{\lambda\rho}\right)$$
$$-\alpha^2\sqrt{2}\,\Omega_{[\mu-}{}^{ab}\bar\epsilon M_{\nu]}{}^{ab},$$

$$\left(\delta_{\alpha^3}+\delta_{\alpha^2\beta}\right)\phi = 3e^\mu{}_a\phi\left(\delta_{\alpha^3}+\delta_{\alpha^2\beta}\right)e_\mu{}^a,$$

$$\left(\delta_{\alpha^3}+\delta_{\alpha^2\beta}\right)\bar\psi_\mu = 2\alpha^2 e^{-1}\phi^3 g_{\mu\rho}\bar\epsilon\mathscr{D}_\lambda(\Omega_+)\left(e\phi^{-3}N^{\lambda\rho}\right),$$

$$\left(\delta_{\alpha^3}+\delta_{\alpha^2\beta}\right)\bar\lambda = \tfrac{1}{4}\sqrt{2}\left(\delta_{\alpha^3}+\delta_{\alpha^2\beta}\right)\bar\psi_\mu\Gamma^\mu. \tag{4.19}$$

With the cancellation mechanism discussed above in mind, it is not surprising that these transformations have exactly the same structure as eq. (3.19). Again the transformation of the zehnbein is determined only up to an (irrelevant) local Lorentz transformation.

In sect. 3 we mentioned another source of new variations of the action which we have not treated thus far. The $O(\alpha^2, \alpha\beta)$ variations of the supergravity fields which we obtained in sect. 3 induce new transformations of the same order of $\Omega_{\mu-}^{ab}$ and ψ^{ab}. When applied to $\mathscr{L}(R)+\mathscr{L}(R^2)$ these transformations give rise to variations of the action of $O(\alpha^3, \alpha^2\beta)$. These variations can be cancelled by using the lemma of sect. 3, and therefore they produce transformations of the supergravity fields of the same form as eq. (4.19), but with different tensors $M_\nu{}^{\lambda\rho}$ and $N^{\lambda\rho}$. We have the following new transformations of $\Omega_{\mu-}^{ab}$ and ψ^{ab}:

$$\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)\Omega_{\mu-}^{ab} = e^{\lambda a}e^{\nu b}\Big[e_{\lambda c}\mathscr{D}_{[\mu}(\omega)\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)e_{\nu]}{}^c + e_{\nu c}\mathscr{D}_{[\lambda}(\omega)\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)e_{\mu]}{}^c$$
$$-e_{\mu c}\mathscr{D}_{[\nu}(\omega)\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)e_{\lambda]}{}^c + \tfrac{3}{2}\sqrt{2}\,\partial_{[\mu}\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)B_{\nu\lambda]}\Big]$$
$$+3\sqrt{2}\,H_\mu{}^{[a}{}_c e^{\rho b]}\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)e_\rho{}^c - \tfrac{1}{4}\bar\psi_{[a}\Gamma_\mu\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)\psi_{b]}$$
$$+3\alpha\left(\left(\delta_\alpha+\delta_\beta\right)\Omega_{[\mu-}{}^{cd}\right)R_{ab]}{}^{cd}(\Omega_-), \tag{4.20}$$

$$\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)\psi^{ab} = 2e^{\mu a}e^{\nu b}\mathscr{D}_{[\mu}(\Omega_+)\left(\delta_{\alpha^2}+\delta_{\alpha\beta}\right)\psi_{\nu]}$$
$$= 3\alpha^2 e^{\mu a}e^{\nu b}\mathscr{D}_{[\mu}(\Omega_+)\left[e^{-1}\phi^3 g_{\nu]\rho}\mathscr{D}_\lambda(\Omega_+)\left(e\phi^{-3}T^{\lambda\rho cd}\right)\Gamma_{cd}\epsilon\right] \tag{4.21}$$

The origin of the different contributions in (4.20) and (4.21) is clear: the new variations are obtained by substituting (3.19) in the definition of $\Omega_{\mu-}^{ab}$ and ψ^{ab}. The last term in eq. (4.20) arises from the variation of the Lorentz Chern–Simons form. The transformations (4.20) and (4.21) contain $\mathscr{D}\epsilon$ terms. It would have been possible to avoid these by a suitable redefinition of $\Omega_{\mu-}^{\ ab}$ and ψ^{ab} in this order in α and β. Such a redefinition would be advantageous for the calculation of the effective action to higher orders, but this falls outside the scope of this paper. To cancel the contribution of (4.20) and (4.25) to the variation of the action additional Noether terms, which involve equations of motion, and variations of the supergravity fields are required. The new variations of the fermions from this source read:

$$
\begin{aligned}
(\delta_{\alpha^3}+\delta_{\alpha^2\beta})\psi_\mu &= 6\alpha^2 e^{-1}\phi^3 g_{\mu\nu}\mathscr{D}_\lambda(\Omega_+)\\
&\quad\times\left[g^{\sigma\lambda}g^{\nu\tau}e\phi^{-3}\mathscr{D}_{[\sigma}(\Omega_+)\left[e^{-1}\phi^3 g_{\tau]\rho}\mathscr{D}_\alpha(\Omega_+)(e\phi^{-3}T^{\alpha\rho ab})\right]\right]\Gamma_{ab}\epsilon\,,\\
(\delta_{\alpha^3}+\delta_{\alpha^2\beta})\lambda &= -\tfrac{1}{4}\sqrt{2}\,\Gamma^\mu(\delta_{\alpha^3}+\delta_{\alpha^2\beta})\psi_\mu\,,
\end{aligned}
\tag{4.22}
$$

and must be added to those given in eq. (4.19). The new transformations of the bosons can be worked out explicitly from eq. (4.20) and the lemma. Since they are not directly relevant for compactification we do not present them.

In the calculations of this section we have not taken into account possible terms which contain explicit $\phi^{-1}\partial\phi$ or H-contributions. It is not hard to see that the most general terms in the action containing $\phi^{-1}\partial\phi$, and not more than bilinear in fermions, are of the form

$$
\phi^{-1}\partial_\mu\phi\left[+g_1\bar{X}_\nu X^{\mu\nu}+g_2\bar{X}X^\mu+g_3T^{\mu\nu\lambda\rho}\left(\operatorname{tr}\bar{\chi}\Gamma_{\nu\lambda\rho}\chi+\bar{\psi}^{ab}\Gamma_{\nu\lambda\rho}\psi_{ab}\right)\right].
\tag{4.23}
$$

In particular, all these terms are bilinear in fermions, and are therefore not directly relevant for analysis of compactification scenarios. With H, a few more index structures than those given in eq. (4.23) are possible, but again all terms are bilinear in fermions. Although a calculation of the coefficients g_i of eq. (4.23), and the corresponding coefficients for the H-terms is in principle possible, it is also extremely cumbersome. As these terms are not required for achieving our main purpose in this paper, we have not attempted to calculate them.

5. Conclusions

In this paper we have obtained the supersymmetric quartic effective action for the heterotic string which follows from the supersymmetrization of the Yang–Mills and Lorentz Chern–Simons forms. In particular this includes all bosonic terms in the action and transformation rules, which have been collected in appendix A. We have thus found all terms which are relevant for compactification.

Both string and sigma-model calculations reveal that the quartic effective action contains another supersymmetric R^4-invariant which does not have a Yang–Mills counterpart. At the tree level the coefficients of these terms contain a factor $\zeta(3)$. The overall coefficient of this R^4-invariant is not determined by requiring supersymmetry and remains a free parameter at this level. It would interesting to see whether other arguments like a suitable compactification or the cancellation of ultra-violet divergences would fix the coefficient.

We expect that the results of this paper will be useful for the study of compactification scenarios of the heterotic string with unbroken supersymmetry. In particular, using our results, it would be interesting to see whether a particular Calabi–Yau manifold is preferred in the compactification.

It is useful to compare our results with the ones following from string amplitude or sigma-model calculations. We find that in our approach the leading term of the quadratic effective action is most naturally given by the Riemann tensor squared. In the other approaches the leading term is usually given by the Gauss–Bonnet combination. The two results differ by terms which are quadratic in the Ricci tensor and scalar. As has been stressed in the introduction, one always has the freedom to redefine the gravity field, thereby introducing such terms in the effective action. To be precise, in order α it is possible to redefine the zehnbein by

$$e_\mu{}^a \to e_\mu{}^a + \alpha\left[a_1 R_\mu{}^a + a_2 e_\mu{}^a R\right], \tag{5.1}$$

with arbitrary coefficients. With a suitable choice of a_1 and a_2, the Gauss–Bonnet combination can be obtained.

We find that the cubic effective action does not contain purely bosonic terms. This is in agreement with string amplitude calculations where the absence of such terms follows from the vanishing of the three-point function.

Finally our results for the quartic effective action are not identical to the ones corresponding to string calculations [5]. It is interesting to consider the correspondence between the two results. In this order, the only redefinition which affects the terms quartic in the Riemann tensor is the following:

$$e_\mu{}^a \to e_\mu{}^a + \alpha^2\left[c_1 R_{\mu\rho}{}^{cd} R^{a\rho cd} + c_2 e_\mu{}^a R_{\nu\rho}{}^{cd} R^{\nu\rho cd}\right]. \tag{5.2}$$

This redefinition gives, when applied to $\mathscr{L}(R^2)$, terms proportional to $\alpha T_{\mu\nu}T^{\mu\nu}$ and αT^2. Therefore we see that the coefficients of these terms can be chosen arbitrarily. However, the redefinition (5.2) also gives rise to additional $\alpha^2 R^3$ terms, proportional to the Einstein tensor, in the effective action. A characteristic feature of the bosonic action at $O(\alpha^3)$ is the absence of a term containing U^2 with $U_{\mu\nu\lambda\rho}$ as given in eq. (4.3). Note that, once a particular parametrization of the bosonic action is chosen, the terms containing the fermions, and the transformation rules, are fixed by supersymmetry.

To summarize, all terms in the effective action we have obtained in this paper follow *uniquely* (modulo the field redefinitions mentioned above) from the supersymmetrization of the Yang–Mills and Lorentz Chern–Simons forms. The structure of these terms agrees with string amplitude calculations at the tree- [4,5] and one-loop [6–8] level. The two results only differ by an overall coefficient which cannot be determined by supersymmetry. Given the uniqueness of our result one would expect that string calculations beyond the one-loop level should always yield the same structure for the quartic effective action. It would be interesting to see whether this result can be understood by looking at the details of higher loop string amplitude calculations.

Let us finally comment on what we expect to happen if one were to try to extend the results of this paper to higher orders in the curvature tensors. All terms in the effective action and transformation rules can be grouped into two kinds. The first kind consists of all terms which follow from an iterative application of the lemma we have derived in sect. 3. We expect that it should be possible to describe the structure of these terms to all orders in an efficient way, thereby possibly unravelling some underlying geometric structure. The other kind contains all terms that follow from a Noether-method calculation. The structure of these terms to all orders is more difficult to predict. Our guess is that in all these terms the symmetry between the supergravity and Yang–Mills fields is manifest. Assuming that this symmetry is indeed present we conjecture that the bosonic terms of the part of the effective action which follows from the supersymmetrization of the Yang–Mills and Lorentz Chern–Simons forms can be parametrized as follows:

$$\mathscr{L}(\text{effective}) + \sum_{n=0,1,2\ldots} \alpha^n\left(a_n R T^n + b_n T^{n+1}\right), \tag{5.3}$$

where R is the Riemann curvature tensor and $T \equiv (\alpha R^2 + \beta F^2)$ (see eq. (4.1)). In this parametrization it is very natural that there is no bosonic $\alpha^2 R^3$ term, since $R_{\mu\nu ab}T^{\mu\nu ab}$ vanishes identically. On the other hand we do expect in the next order to find $\alpha^4 R^5$ terms since, e.g. $R_{\mu\nu}{}^{ab}T^{\mu\nu cd}T_{abcd} \neq 0$. Such terms will for instance be needed for the cancellation of terms of the form $\alpha^2 \epsilon RT\partial X$ with $X \equiv (\alpha R\psi + \beta F\chi)$ (see eq. (4.2)). They arise for instance from the variation $\delta\psi_\mu \sim \alpha^2\,\partial(RT)\epsilon$ (see eq. (4.19)) in the Noether terms $X\psi_\mu$ of the $\alpha R^2 + \beta F^2$-action.

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Appendix A

THE BOSONIC ACTION AND TRANSFORMATION RULES

In this appendix we gather all bosonic terms in the action and in the supersymmetry transformation rules, in all the orders in α and β that we have considered in this paper. For easy reference we try to make this appendix self-contained, and therefore repeat some of the definitions that are required to understand the different contributions.

The bosonic terms in the action are

$$\mathscr{L}= e\phi^{-3}\Big[-\tfrac{1}{2}R(\omega(e)) - \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} + \tfrac{9}{2}\big(\phi^{-1}\partial_\mu\phi\big)^2$$
$$+\tfrac{1}{4}T + \tfrac{3}{2}\alpha T^{\mu\nu\lambda\rho}T_{\mu\nu\lambda\rho} + \tfrac{1}{2}\alpha T^{\mu\nu}T_{\mu\nu}\Big]. \tag{A.1}$$

Our conventions for the Riemann tensor are

$$R_{\mu\nu}{}^{ab}(\omega) \equiv 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - \omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b}, \qquad R(\omega) \equiv e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}(\omega). \tag{A.2}$$

The tensors T are defined as:

$$T_{\mu\nu\lambda\rho} \equiv \alpha R_{[\mu\nu}{}^{ab}(\Omega_-)R_{\lambda\rho]}{}^{ab}(\Omega_-) + \beta\,\mathrm{tr}\,F_{[\mu\nu}F_{\lambda\rho]},$$
$$T_{\mu\nu} \equiv \alpha R_{\mu\lambda}{}^{ab}(\Omega_-)R^{\lambda}{}_{\nu}{}^{ab}(\Omega_-) + \beta\,\mathrm{tr}\,F_{\mu\lambda}F^{\lambda}{}_{\nu}, \qquad T \equiv T_\mu^\mu. \tag{A.3}$$

In T one finds the Riemann tensor of Ω_-, which is defined as

$$\Omega_{\mu\pm}^{ab} \equiv \omega_\mu{}^{ab} \pm \tfrac{3}{2}\sqrt{2}\,H_\mu{}^{ab}. \tag{A.4}$$

In (A.1) and (A.4) $H_{\mu\nu\rho}$ is the curl of the antisymmetric gauge field $B_{\mu\nu}$, and includes the Lorentz and Yang–Mills Chern–Simons forms:

$$H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} - \alpha\sqrt{2}\,\Big\{\Omega_{[\mu-}^{ab}\,\partial_\nu\Omega_{\rho]-}^{ab} - \tfrac{2}{3}\Omega_{[\mu-}^{ab}\Omega_{\nu-}^{ac}\Omega_{\rho]-}^{cb}\Big\}$$
$$-\beta\sqrt{2}\,\mathrm{tr}\Big\{A_{[\mu}\partial_\nu A_{\rho]} - \tfrac{1}{3}A_{[\mu}A_\nu A_{\rho]}\Big\}. \tag{A.5}$$

The R-action and the $O(\alpha,\beta)$ action are discussed in sect. 2 and appendix B, the $O(\alpha^3, \alpha^2\beta, \alpha\beta^2)$ action in sect. 4.

The relevant transformation rules are those of the supergravity fields ψ_μ and λ, and of the Yang–Mills field χ. We write $\delta_{\alpha^n}(\delta_{\beta^n})$ for variations of order $\alpha^n(\beta^n)$, while δ_0 corresponds to the terms independent of α and β. We have the following supersymmetry transformations:

$$\delta_0\psi_\mu = \Big(\partial_\mu - \tfrac{1}{4}\Gamma^{ab}\Omega_{\mu+}^{ab}\Big)\epsilon, \qquad \delta_0\lambda = -\tfrac{3}{8}\sqrt{2}\,\big(\phi^{-1}\not{\partial}\phi\big)\epsilon + \tfrac{1}{8}\Gamma^{abc}\epsilon H_{abc}; \tag{A.6}$$

$$\delta_0\chi = -\tfrac{1}{4}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon; \tag{A.7}$$

$$\big(\delta_{\alpha^2} + \delta_{\alpha\beta}\big)\psi_\mu = \tfrac{3}{2}\alpha e^{-1}\phi^3 g_{\mu\rho}\mathscr{D}_\lambda(\Omega_+)\big[e\phi^{-3}T^{\lambda\rho ab}\Gamma_{ab}\big]\epsilon,$$
$$\big(\delta_{\alpha^2} + \delta_{\alpha\beta}\big)\lambda = -\tfrac{1}{4}\sqrt{2}\,\Gamma^\mu\big(\delta_{\alpha^2} + \delta_{\alpha\beta}\big)\psi_\mu; \tag{A.8}$$

$$\delta_{\alpha\beta}\chi = 0\,; \tag{A.9}$$

$$\begin{aligned}(\delta_{\alpha^3} + \delta_{\alpha^2\beta})\psi_\mu = {} & 6\alpha^2 e^{-1}\phi^3 g_{\mu\nu}\mathscr{D}_\lambda(\Omega_+)\\ & \times\left[g^{\sigma\lambda}g^{\nu\tau}e\phi^{-3}\times\mathscr{D}_{[\sigma}(\Omega_+)\left[e^{-1}\phi^3 g_{\tau]\rho}\mathscr{D}_\alpha(\Omega_+)(e\phi^{-3}T^{\alpha\rho ab})\right]\right]\Gamma_{ab}\epsilon\\ & +2\alpha^2 e^{-1}\phi^3 g_{\mu\rho}\mathscr{D}_\lambda(\Omega_+)\left[e\phi^{-3}\left\{\tfrac{1}{32}T\Gamma^{\sigma\tau}R_{\sigma\tau}{}^{\lambda\rho}(\Omega_-)\right.\right.\\ & \left.\left.-\tfrac{1}{2}T^{\sigma\tau}\Gamma^\nu\Gamma_\sigma R_{\tau\nu}{}^{\lambda\rho}(\Omega_-)+\tfrac{1}{64}\Gamma^{\sigma\tau}\Gamma^{\alpha\beta\gamma\delta}T_{\alpha\beta\gamma\delta}R_{\sigma\tau}{}^{\lambda\rho}(\Omega_-)\right\}\right]\epsilon\,,\end{aligned}$$

$$(\delta_{\alpha^3}+\delta_{\alpha^2\beta})\lambda = -\tfrac{1}{4}\sqrt{2}\,\Gamma^\mu(\delta_{\alpha^3}+\delta_{\alpha^2\beta})\psi_\mu\,; \tag{A.10}$$

$$(\delta_{\alpha^2\beta}+\delta_{\alpha\beta^2})\chi = -\alpha\beta\left[\tfrac{1}{32}T\Gamma^{\sigma\tau}F_{\sigma\tau}-\tfrac{1}{2}T^{\sigma\tau}\Gamma^\nu\Gamma_\sigma F_{\tau\nu}+\tfrac{1}{64}\Gamma^{\sigma\tau}\Gamma^{\alpha\beta\gamma\delta}T_{\alpha\beta\gamma\delta}F_{\sigma\tau}\right]\epsilon\,. \tag{A.11}$$

In sect. 2 and appendix B we discuss the specific form of eqs. (A.6) and (A.7). The variations of $O(\alpha^2, \alpha\beta)$ are derived in sect. 3. Finally the origin of eqs. (A.10) and (A.11) can be found in sect. 4.

Appendix B

THE COMPLETE F^2- AND R^2-ACTION

In this appendix we will add some details to the presentation of R^2-actions in sect. 2. The starting point must be the complete transformation rules and invariant action for the coupled supergravity and Yang–Mills fields. With our choice of basis, which differs from the one employed in ref. [22], the transformation rules are in lowest order in β:

$$\begin{aligned}&\delta_0 e_\mu{}^a = \tfrac{1}{2}\bar\epsilon\Gamma^a\psi_\mu\,, && \delta_0\psi_\mu = \left(\partial_\mu - \tfrac{1}{4}\Omega_{\mu+}^{\;ab}\Gamma_{ab}\right)\epsilon + \tfrac{1}{2}\sqrt{2}\left(\epsilon\bar\psi_\mu\lambda - \psi_\mu\bar\epsilon\lambda + \Gamma^a\lambda\bar\psi_\mu\Gamma_a\epsilon\right),\\ &\delta_0 B_{\mu\nu} = \tfrac{1}{2}\sqrt{2}\,\bar\epsilon\Gamma_{[\mu}\psi_{\nu]}\,, && \delta_0\lambda = -\tfrac{3}{8}\sqrt{2}\,\phi^{-1}\not{D}\phi\epsilon + \tfrac{1}{8}\Gamma^{abc}\epsilon\left(\hat H_{abc} - \tfrac{1}{24}\sqrt{2}\,\bar\lambda\Gamma_{abc}\lambda\right),\\ &\phi^{-1}\delta_0\phi = -\tfrac{1}{3}\sqrt{2}\,\bar\epsilon\lambda\,; && \end{aligned} \tag{B.1}$$

$$\delta_0 A_\mu = \tfrac{1}{2}\bar\epsilon\Gamma_\mu\chi\,, \qquad \delta_0\chi = -\tfrac{1}{4}\Gamma^{ab}\epsilon\hat F_{ab} + \tfrac{1}{2}\sqrt{2}\,(\epsilon\bar\chi\lambda - \chi\bar\epsilon\lambda + \Gamma^a\lambda\bar\chi\Gamma_a\epsilon)\,. \tag{B.2}$$

There are modifications of $O(\beta)$ to these transformation rules:

$$\begin{aligned}&\delta_\beta\psi_\mu = (1/192)\beta\Gamma^{abc}\Gamma_\mu\epsilon\,\mathrm{tr}\,\bar\chi\Gamma_{abc}\chi\,, \qquad \delta_\beta B_{\mu\nu} = -\beta\sqrt{2}\,\mathrm{tr}\{A_{[\mu}\delta_0 A_{\nu]}\}\,,\\ &\delta_\beta\lambda = (1/384)\beta\sqrt{2}\,\Gamma^{abc}\epsilon\,\mathrm{tr}\,\bar\chi\Gamma_{abc}\chi\,.\end{aligned} \tag{B.3}$$

Derivatives $\mathscr{D}$ are covariant with respect to Lorentz and Yang–Mills gauge transformations, while D is also supercovariant. On curvatures we indicate supercovariance with a hat. The combination $\Omega_{\mu\pm}^{ab}$ is given by

$$\Omega_{\mu\pm}^{ab} = \omega_\mu{}^{ab}(e,\psi) \pm \tfrac{3}{2}\sqrt{2}\,\hat{H}_{\mu ab}, \tag{B.4}$$

where $\omega(e,\psi)$ is the solution of $D_{[\mu}(\omega)e^a{}_{\nu]} = 0$. The supersymmetry transformation of $\omega(e,\psi)$ reads

$$\delta_0\omega_\mu{}^{ab}(e,\psi) = \tfrac{1}{4}\bar{\epsilon}\Gamma_\mu\psi^{ab} + \tfrac{1}{2}\bar{\epsilon}\Gamma^{[a}\psi_\mu{}^{b]} + \tfrac{3}{4}\sqrt{2}\,\bar{\epsilon}\Gamma^c\psi_\mu\hat{H}^{abc},$$

$$\delta_\beta\omega_\mu{}^{ab}(e,\psi) = (1/192)\beta\bar{\epsilon}\Gamma^{[a}\Gamma_{\mathrm{def}}\Gamma^{b]}\psi_\mu \operatorname{tr}\bar{\chi}\Gamma^{\mathrm{def}}\chi\,. \tag{B.5}$$

The action $\mathscr{L}(R) + \mathscr{L}(F^2)$, which is invariant under (B.1)–(B.3), is given by

$$\begin{aligned}\mathscr{L}(R) = e\phi^{-3}\Big\{ & -\tfrac{1}{2}R(\omega(e)) - \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} + \tfrac{9}{2}\left(\phi^{-1}\partial_\mu\phi\right)^2 \\ & -\tfrac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\mathscr{D}_\nu(\omega(e))\psi_\rho + 2\sqrt{2}\,\bar{\lambda}\Gamma^{\mu\nu}\mathscr{D}_\mu(\omega(e))\psi_\nu + 4\bar{\lambda}\not{\mathscr{D}}(\omega(e))\lambda \\ & +3\sqrt{2}\,\bar{\psi}_\mu\Gamma^\nu\Gamma^\mu\lambda\left(\phi^{-1}\partial_\nu\phi\right) - \tfrac{3}{2}\bar{\psi}_\mu\Gamma^\mu\psi_\nu\left(\phi^{-1}\partial^\nu\phi\right) \\ & +\tfrac{1}{16}\sqrt{2}\,H^{\rho\sigma\tau}\left[\bar{\psi}_\mu\Gamma^{[\mu}\Gamma_{\rho\sigma\tau}\Gamma^{\nu]}\psi_\nu + 4\sqrt{2}\,\bar{\psi}_\mu\Gamma^\mu{}_{\rho\sigma\tau}\lambda - 8\bar{\lambda}\Gamma_{\rho\sigma\tau}\lambda\right] \\ & +\tfrac{1}{96}\bar{\psi}^\mu\Gamma^{abc}\psi_\mu\left[\bar{\lambda}\Gamma_{abc}\lambda + \tfrac{1}{2}\sqrt{2}\,\bar{\lambda}\Gamma_{abc}\Gamma\cdot\psi - \tfrac{1}{4}\bar{\psi}^\nu\Gamma_{abc}\psi_\nu - \tfrac{1}{8}\bar{\psi}\cdot\Gamma\Gamma_{abc}\Gamma\cdot\psi\right]\Big\}.\end{aligned} \tag{B.6}$$

$$\begin{aligned}\mathscr{L}(F^2) = e\phi^{-3}\beta\operatorname{tr}\Big\{ & -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\bar{\chi}\not{\mathscr{D}}(\omega(e,\psi),A)\chi \\ & -\tfrac{1}{8}\bar{\chi}\Gamma^\mu\Gamma^{ab}\left(F_{ab} + \hat{F}_{ab}\right)\left(\psi_\mu + \tfrac{1}{3}\sqrt{2}\,\Gamma_\mu\lambda\right) + \tfrac{1}{16}\sqrt{2}\,\bar{\chi}\Gamma^{abc}\chi\hat{H}_{abc} \\ & -[1/(16\times 24)]\sqrt{2}\,\bar{\chi}\Gamma^{abc}\chi\bar{\psi}_\mu(4\Gamma_{abc}\Gamma^\mu + 3\Gamma^\mu\Gamma_{abc})\lambda \\ & +(1/96)\bar{\chi}\Gamma^{abc}\chi\bar{\lambda}\Gamma_{abc}\lambda - [1/(16\times 24)]\,\beta\bar{\chi}\Gamma^{abc}\chi\operatorname{tr}\bar{\chi}\Gamma_{abc}\chi\Big\}.\end{aligned} \tag{B.7}$$

The four-fermion terms in eq. (B.6) were obtained from the requirement of supercovariance of the λ and ψ_μ equations of motion. Note that it is not possible to have $\lambda^3\psi_\mu$ or λ^4 terms due to the ten-dimensional identity

$$\bar{\lambda}\Gamma^{abc}\lambda\bar{\lambda}\Gamma_{ab} = 0\,. \tag{B.8}$$

The terms with four gravitinos are not affected by our redefinitions, and therefore coincide with those given in ref. [22].

It is useful to have the precise definitions of supercovariant derivatives of various fields and their variations at hand. We include here all $O(\beta)$ modifications due to the coupling to Yang–Mills. First the definitions:

$$\phi^{-1}D_\mu\phi = \phi^{-1}\partial_\mu\phi + \tfrac{1}{3}\sqrt{2}\,\bar\psi_\mu\lambda\,, \tag{B.9}$$

$$\begin{aligned} D_\mu(\omega)\lambda = {} & \mathscr{D}_\mu(\omega)\lambda + \tfrac{3}{8}\sqrt{2}\,\phi^{-1}\not{D}\phi\psi_\mu - \tfrac{1}{8}\Gamma^{abc}\psi_\mu\left(\hat H_{abc} - \tfrac{1}{24}\sqrt{2}\,\bar\lambda\Gamma_{abc}\lambda\right) \\ & - \tfrac{1}{384}\beta\sqrt{2}\,\Gamma^{abc}\psi_\mu\,\mathrm{tr}\,\bar\chi\Gamma_{abc}\chi\,, \end{aligned} \tag{B.10}$$

$$\hat H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} - \tfrac{1}{4}\sqrt{2}\,\bar\psi_{[\mu}\Gamma_\nu\psi_{\rho]} - \beta\sqrt{2}\ \mathrm{tr}\left\{A_{[\mu}\partial_\nu A_{\rho]} - \tfrac{1}{3}A_{[\mu}A_\nu A_{\rho]}\right\}, \tag{B.11}$$

$$\begin{aligned} \psi_{\mu\nu} = {} & \mathscr{D}_\mu(\Omega_+)\psi_\nu - \mathscr{D}_\nu(\Omega_+)\psi_\mu - \tfrac{1}{2}\sqrt{2}\left(\psi_\mu\bar\psi_\nu\lambda - \psi_\nu\bar\psi_\mu\lambda - \Gamma^a\lambda\bar\psi_\mu\Gamma_a\psi_\nu\right) \\ & - \tfrac{1}{96}\beta\Gamma^{abc}\Gamma_{[\mu}\psi_{\nu]}\,\mathrm{tr}\,\bar\chi\Gamma_{abc}\chi\,, \end{aligned} \tag{B.12}$$

$$\hat F_{\mu\nu} = F_{\mu\nu} - \bar\psi_{[\mu}\Gamma_{\nu]}\chi\,, \tag{B.13}$$

$$D_\mu(\omega, A)\chi = \mathscr{D}_\mu(\omega, A)\chi + \tfrac{1}{4}\Gamma^{ab}\psi_\mu\hat F_{ab} - \tfrac{1}{2}\sqrt{2}\left(\psi_\mu\bar\chi\lambda - \chi\bar\psi_\mu\lambda + \Gamma^a\lambda\bar\chi\Gamma_a\psi_\mu\right). \tag{B.14}$$

For some of these derivatives and curvatures in (B.9)–(B.14) we need the supersymmetry transformation as well. Again we distinguish between δ_0, the lowest order transformation rule, and δ_β, the modifications due to the Einstein–Yang–Mills coupling:

$$\delta_0\left(\phi^{-1}D_a\phi\right) = -\tfrac{1}{3}\sqrt{2}\,\bar\epsilon D_a(\Omega_+)\lambda\,, \qquad \delta_0\hat H_{abc} = -\tfrac{1}{4}\sqrt{2}\,\bar\epsilon\Gamma_{[a}\psi_{bc]} \qquad \text{(B.15), (B.16)}$$

$$\delta_0\psi_{ab} = -\tfrac{1}{4}\Gamma^{cd}\epsilon\hat R_{cd}{}^{ab}(\Omega_-) + \tfrac{1}{2}\sqrt{2}\left(\epsilon\bar\psi_{ab}\lambda - \psi_{ab}\bar\epsilon\lambda + \Gamma^c\lambda\bar\psi_{ab}\Gamma_c\epsilon\right), \tag{B.17}$$

$$\delta_0\hat F_{ab} = -\bar\epsilon\Gamma_{[a}D_{b]}(\Omega_+, A)\chi \tag{B.18}$$

$$\delta_\beta\left(\phi^{-1}D_a\phi\right) = -[1/(3\times 192)]\,\beta\sqrt{2}\,\bar\epsilon\Gamma_\mu\Gamma^{abc}\lambda\,\mathrm{tr}\,\bar\chi\Gamma_{abc}\chi\,, \tag{B.19}$$

$$\delta_\beta\hat H_{abc} = -\tfrac{1}{2}\beta\sqrt{2}\,\bar\epsilon\Gamma_{[a}\mathrm{tr}\,\chi\hat F_{bc]}\,, \tag{B.20}$$

$$\begin{aligned} \delta_\beta\psi_{ab} = \beta\ \mathrm{tr}\Big\{ & \tfrac{3}{4}\Gamma^{cd}\epsilon\hat F_{[ab}\hat F_{cd]} + \tfrac{1}{48}\Gamma^{cde}\Gamma_{[a}\epsilon\bar\chi\Gamma_{cde}D_{b]}(\Omega_+, A)\chi \\ & + (1/256)\sqrt{2}\,\Gamma^{cde}\Gamma_{[a}\Gamma^{gh}\epsilon\hat H_{b]gh}\bar\chi\Gamma_{cde}\chi \\ & + [1/(96\times 96)]\,\beta\Gamma^{cde}\Gamma_{[a}\Gamma^{fgh}\Gamma_{b]}\epsilon\bar\chi\Gamma_{cde}\chi\,\mathrm{tr}\,\bar\chi\Gamma_{fgh}\chi\Big\}, \end{aligned} \tag{B.21}$$

$$\delta_\beta\hat F_{ab} = (1/192)\beta\bar\epsilon\Gamma_{[a}\Gamma^{cde}\Gamma_{b]}\chi\,\mathrm{tr}\,\bar\chi\Gamma_{cde}\chi \tag{B.22}$$

Let us briefly discuss the transformation of ψ_{ab}. The variation of the gravitino curvature $\psi_{\mu\nu}$ contains, as we can see from eq. (B.1), the commutator

$$[\mathscr{D}_\mu(\Omega_+), \mathscr{D}_\nu(\Omega_+)]\epsilon = -\tfrac{1}{4}\Gamma^{cd}\epsilon R_{\mu\nu}{}^{cd}(\Omega_+). \tag{B.23}$$

To obtain eq. (B.17) one needs to use the identity

$$R_{abcd}(\Omega_+) - R_{cdab}(\Omega_-) = 6\sqrt{2}\mathscr{D}_{[a}(\omega)H_{bcd]} = -3\beta \operatorname{tr} F_{[ab}F_{cd]}. \tag{B.24}$$

The Bianchi identity for the curvature H does not vanish due to the presence of the Yang–Mills Chern–Simons term.

It is both useful and instructive to obtain the supersymmetry algebra from (B.1)–(B.2). The commutator of two supersymmetry transformations reads

$$\begin{aligned}[\delta(\epsilon_1), \delta(\epsilon_2)] &= \delta_{\mathrm{P}}(\xi^\mu) + \delta_{\mathrm{Q}}(-\xi^\mu\psi_\mu) + \delta_{\mathrm{L}}\left(-\xi^\mu\Omega_{\mu-}^{\ ab}\right) + \delta_{\mathrm{YM}}(-\xi^\mu A_\mu) \\ &\quad + \delta_{\mathrm{M}}\left(-\tfrac{1}{2}\sqrt{2}\,\xi_\mu - \xi^\nu B_{\nu\mu}\right) + \delta_{\mathrm{Q}}(\epsilon_3) + \delta_{\mathrm{L}}(\Lambda^{ab}),\end{aligned} \tag{B.25}$$

where

$$\begin{aligned}\xi^\mu &= \tfrac{1}{2}\bar\epsilon_2\Gamma^\mu\epsilon_1, \\ \epsilon_3 &= -\tfrac{7}{16}\sqrt{2}\,\bar\epsilon_2\Gamma^a\epsilon_1\Gamma_a\lambda + [1/(32\times 120)]\sqrt{2}\,\bar\epsilon_2\Gamma^{a_1\cdots a_5}\epsilon_1\Gamma_{a_1\dots a_5}\lambda, \\ \Lambda^{ab} &= \tfrac{1}{192}\beta\bar\epsilon_2\Gamma^{[a}\Gamma_{cde}\Gamma^{b]}\epsilon_1 \operatorname{tr}\bar\chi\Gamma^{cde}\chi.\end{aligned} \tag{B.26}$$

On the right-hand side of (B.25) we encounter all gauge transformations of the $d = 10$ super-Yang–Mills theory: δ_{P}, δ_{Q}, δ_{L}, δ_{YM}, and δ_{M} correspond respectively to general coordinate, supersymmetry, Lorentz, Yang–Mills and antisymmetric tensor gauge transformations. Note in particular that in lowest order in β the Lorentz transformation and the Yang–Mills transformation occur with the same structure in eq. (B.25). Thus the combination $\Omega_{\mu-}^{\ ab}$ and the Yang–Mills gauge field A_μ play the same role in the supersymmetry algebra, which makes the symmetry between the Yang–Mills and Lorentz groups in the Einstein–Yang–Mills theory manifest.

From eq. (B.26) we can also understand some of the simplifications which occur in eq. (B.1) as compared to the form given in ref. [22]. There is no λ^2-dependent Lorentz transformation in the algebra, which of course is related to the fact that ψ_μ does not transform to λ^2. Also note that the terms bilinear in fermions in $\delta_0\psi_\mu$ and $\delta_0\chi$ have the same form. This is implied by the fact that they are responsible for realising the λ-dependent Q-transformation in the algebra on $e_\mu{}^a$ and A_μ, respectively, and this happens in exactly the same manner.

Let us now exploit the symmetry between Lorentz and Yang–Mills groups. The algebraic structure we have obtained makes it possible to extract an SO(9,1)

Yang–Mills multiplet from the supergravity fields. The role of the SO(9, 1) gauge field must be played by the combination $\Omega_{\mu-}^{\ ab}$. Its transformation rule can be read off from eqs. (B.5) and (B.16):

$$\delta_0 \Omega_{\mu-}^{\ ab} = \tfrac{1}{2}\bar{\epsilon}\Gamma_\mu \psi^{ab}. \tag{B.27}$$

The fermion of the SO(9, 1) Yang–Mills multiplet is therefore the covariant gravitino curvature ψ_{ab}. Indeed, in eq. (B.17) we see that its transformation rule, with this identification, is identical to that of χ in eq. (B.2). Therefore, we may use the knowledge of the F^2-invariant to write down an R^2-invariant as well. To this end, we introduce a dimensionful constant α, and write:

$$\begin{aligned}\mathscr{L}(R^2) = e\phi^{-3}\alpha\Big\{ & -\tfrac{1}{4}R^{\mu\nu ab}(\Omega_-)R_{\mu\nu}^{\ \ ab}(\Omega_-) - \tfrac{1}{2}\bar{\psi}^{ab}\not{\mathscr{D}}(\omega(e,\psi),\Omega_-)\psi_{ab} \\ & -\tfrac{1}{8}\bar{\psi}^{ab}\Gamma^\mu\Gamma^{\nu\rho}\big(R_{\nu\rho}^{\ \ ab}(\Omega_-) + \hat{R}_{\nu\rho}^{\ \ ab}(\Omega_-)\big)\big(\psi_\mu + \tfrac{1}{3}\sqrt{2}\,\Gamma_\mu\lambda\big) \\ & + \tfrac{1}{16}\sqrt{2}\,\bar{\psi}^{ab}\Gamma^{\mu\nu\rho}\psi_{ab}\hat{H}_{\mu\nu\rho} - [1/(16\times 24)]\sqrt{2}\,\bar{\psi}^{ab}\Gamma^{cde}\psi_{ab}\bar{\psi}_\mu(4\Gamma_{cde}\Gamma^\mu + 3\Gamma^\mu\Gamma_{cde})\lambda \\ & + \tfrac{1}{96}\bar{\psi}^{ab}\Gamma^{cde}\psi_{ab}\bar{\lambda}\Gamma_{cde}\lambda - [1/(16\times 24)]\,\alpha\bar{\psi}^{ab}\Gamma^{fgh}\psi_{ab}\bar{\psi}^{cd}\Gamma_{fgh}\psi_{cd}\Big\}. \end{aligned} \tag{B.28}$$

Let us first set $\beta = 0$, and discuss the invariance of the action $\mathscr{L}(R) + \mathscr{L}(R^2)$. From eq. (B.3) we see that the invariance requires new transformations of the supergravity fields of $O(\alpha)$:

$$\begin{aligned} &\delta_\alpha\psi_\mu = \tfrac{1}{192}\alpha\Gamma^{cde}\Gamma_\mu\epsilon\bar{\psi}^{ab}\Gamma_{cde}\psi_{ab}, \qquad \delta_\alpha B_{\mu\nu} = -\alpha\sqrt{2}\,\Omega_{[\mu-}^{\ \ ab}\delta_0\Omega_{\nu]-}^{\ \ ab}, \\ &\delta_\alpha\lambda = \tfrac{1}{384}\alpha\sqrt{2}\,\Gamma^{cde}\epsilon\bar{\psi}^{ab}\Gamma_{cde}\psi_{ab}. \end{aligned} \tag{B.29}$$

Furthermore, the curvature H must everywhere be modified with the appropriate Chern–Simons form, which in the present case means:

$$H_{\mu\nu\rho} \to H_{\mu\nu\rho} - \alpha\sqrt{2}\left\{\Omega_{[\mu-}^{\ \ ab}\partial_\nu\Omega_{\rho-]}^{\ \ ab} - \tfrac{2}{3}\Omega_{[\mu-}^{\ \ ab}\Omega_{\nu-}^{\ \ ac}\Omega_{\rho]-}^{\ \ cb}\right\}. \tag{B.30}$$

Are these modifications sufficient for exact invariance? They would be, were it not for the fact that $\Omega_{\mu-}^{\ ab}$ and ψ^{ab} are made up of fields of the supergravity multiplet, and that therefore modifications of their transformation rules are induced by eq. (B.29). That this will happen we can already see in eqs. (B.5), (B.20) and (B.21). All these modifications now occur with coefficient α as well, and imply:

$$\begin{aligned} \delta_\alpha\Omega_{\mu-}^{\ ab} &= \alpha\left\{\tfrac{3}{2}\bar{\epsilon}\Gamma_{[\mu}\hat{R}_{ab]}^{\ \ cd}\psi_{cd} + \tfrac{1}{192}\bar{\epsilon}\Gamma^{[a}\Gamma_{fgh}\Gamma^{b]}\psi_\mu\bar{\psi}^{cd}\Gamma^{fgh}\psi_{cd}\right\} \\ \delta_\alpha\psi_{ab} &= \alpha\Big\{\tfrac{3}{4}\Gamma^{cd}\epsilon\hat{R}_{[ab}^{\ \ ef}\hat{R}_{cd]}^{\ \ ef} + \tfrac{1}{48}\Gamma^{fgh}\Gamma_{[a}\epsilon\bar{\psi}^{cd}\Gamma_{fgh}D_{b]}(\Omega_+,\Omega_-)\psi_{cd} \\ &\quad + \tfrac{1}{256}\sqrt{2}\,\Gamma^{fgh}\Gamma_{[a}\Gamma^{mn}\epsilon\hat{H}_{b]mn}\bar{\psi}^{cd}\Gamma_{fgh}\psi_{cd} \\ &\quad + [1/(96\times 96)]\,\alpha\Gamma^{hkl}\Gamma_{[a}\Gamma^{mnp}\Gamma_{b]}\epsilon\bar{\psi}^{cd}\Gamma_{hkl}\psi_{cd}\bar{\psi}^{ef}\Gamma_{mnp}\psi_{ef}\Big\}. \end{aligned} \tag{B.31}$$

So the induced modifications (B.31) mark the difference between a true Yang–Mills multiplet and the SO(9, 1) multiplet made up by $\Omega_{\mu-}^{\;ab}$ and ψ_{ab}. Since eq. (B.31) is in fact the *only* difference, we can now state very precisely in what sense eq. (B.28) is invariant. The only transformations which break the exact invariance are due to eq. (B.31) applied to the action $\mathscr{L}(R)+\mathscr{L}(R^2)$. In eq. (B.28) there is an explicit dependence on $\Omega_{\mu-}^{\;ab}$ and ψ^{ab}, and in eqs. (B.6) and (B.28) also an implicit dependence on $\Omega_{\mu-}^{\;ab}$ due to the Chern–Simons form in $H_{\mu\nu\rho}$. These variations of the action are all of $O(\alpha^2)$.

This result trivially extends to the case $\beta \neq 0$, where we consider the full action $\mathscr{L}(R)+\mathscr{L}(F^2)+\mathscr{L}(R^2)$. Again the only variations which break invariance are the transformations of $\Omega_{\mu-}^{\;ab}$ and ψ_{ab} induced by the modifications of the basic fields. These can again be read off from eqs. (B.5), (B.20) and (B.21) for the contribution of the Yang–Mills sector, and equal (B.31) for the Lorentz sector.

The new variations, which we must consider in the construction of invariants of higher order in α and β, have two sources. First there are terms which arise from the new variation of $B_{\mu\nu}$ given in eqs. (B.3) and (B.29). They manifest themselves in δH, and also in $\delta\psi^{ab}$ through the mechanism indicated in eqs. (B.23) and (B.24). Then there are terms which arise from the new variation of ψ_μ and λ. These are bilinear in Fermi fields, and complicated. A generalization of our results to higher orders in α and β would be extremely complicated, not to say practically impossible, if such terms were to be taken into account. This is the main reason that we limit ourselves, in the body of this paper, to leading terms, which excludes the bilinear fermion terms in the variations of the fields. This then implies that we have no control over four-fermion terms in the action, and cannot discuss them either.

The relevant induced transformations which generate the higher-order invariants are then

$$\delta\Omega_{\mu-}^{\;ab} = \tfrac{3}{2}\alpha\bar{\epsilon}\Gamma_{[\mu}\hat{R}_{ab]}^{\;\;cd}\psi_{cd} + \tfrac{3}{2}\beta\bar{\epsilon}\Gamma_{[\mu}\operatorname{tr}\chi\hat{F}_{ab]},$$

$$\delta\psi_{ab} = \tfrac{3}{4}\alpha\Gamma^{cd}\epsilon\hat{R}_{[ab}^{\;\;ef}\hat{R}_{cd]}^{\;\;ef} + \tfrac{3}{4}\beta\Gamma^{cd}\epsilon\operatorname{tr}\hat{F}_{[ab}\hat{F}_{cd]} \tag{B.32}$$

They arise directly from the Chern–Simons terms.

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